Theory of Smith Forms for Bivariate Polynomial Matrices*

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Abstract This paper studies the equivalence theory between bivariate polynomial matrices and their Smith forms. For a class of bivariate polynomial matrices, by leveraging the special form of the greatest common divisor of the maximal minors of the matrix, the authors construct a homomorphism from the bivariate polynomial ring to a Euclidean domain. Subsequently, by applying Gaussian elimination, the matrix can be reduced to its Smith form. Consequently, the authors establish that the necessary and sufficient condition for such a type of matrix to be equivalent to its Smith form is that the reduced minors of each order generate the unit ideal.

Keywords Bivariate polynomial matrices, matrix equivalence, reduced minors, Smith forms.

1 Introduction

Multivariate polynomial matrices serve as a fundamental representational framework for characterizing diverse multidimensional systems encountered in circuits, control, signal processing, such as dynamical control systems and delay-differential systems (see [1–4] and the

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references therein). The algebraic properties of multidimensional systems can be systematically established through rigorous investigation of their corresponding multivariate polynomial matrices. Owing to the structural simplicity and information-preserving characteristics inherent in the Smith form, mathematicians and engineers are particularly concerned about the equivalence problem between multivariate polynomial matrices and their Smith forms.

Since univariate polynomial rings are Euclidean domains, univariate polynomial matrices are always equivalent to their Smith forms[5]. However, this fact may not hold in the multivariate case. For instance, Frost and Storey[6] provided a counter-example to illustrate that a bivariate polynomial matrix is not equivalent to its Smith form. Therefore, researchers began to explore the conditions under which a multivariate polynomial matrix is equivalent to its Smith form.

Since the 1970s, there have been some results on the equivalence between bivariate polynomial matrices and their Smith forms (see, e.g., [7–9]). Meanwhile, for cases involving more than two variables, researchers have primarily focused on investigating matrices with special forms. For example, Lin, et al.[10] proved that a square matrix $F \in K[x_1, \cdots, x_n]^{l\times l}$ with $\det(F) = x_1 - f(x_2, \cdots, x_n)$ is equivalent to its Smith form $S = \mathrm{diag}\{1, \cdots, 1, \det(F)\}$. Building on the main result presented in [10], Boudellioua[11] implemented the corresponding equivalence algorithm in the computer algebra system Maple, and constructed two unimodular matrices to reduce F to S. Subsequently, Li, et al. generalized the conclusion in [10] to the case of $\det(F) = (x_1 - f(x_2, \cdots, x_n))^t$ through a series of works (see, e.g., [12–14]). In addition, Liu, et al.[15] studied the equivalence problem concerning an upper triangular matrix $F \in K[x_1, \cdots, x_n]^{l\times l}$ with diagonal entries being $x_1 - f(x_2, \cdots, x_n) - c_1, \cdots, x_1 - f(x_2, \cdots, x_n) - c_l$ and its Smith form $S = \mathrm{diag}\{1, \cdots, 1, \det(F)\}$, and proved that F is necessarily equivalent to S, where $c_1, \cdots, c_l \in K$ are pairwise different. In 2019, Li, et al.[16] investigated the equivalence problem of another type of bivariate polynomial matrices. Let $F \in K[x, y]^{l\times l}$ with $\det(F) = p$, where $p \in K[x]$ is an irreducible polynomial. Then F is equivalent to its Smith form $\mathrm{diag}\{1, \cdots, 1, p\}$. Zheng, et al.[17] extended the main result in [16] to the case of $\det(F) = p^t$, and proposed that the necessary and sufficient condition for F to be equivalent to its Smith form $\mathrm{diag}\{p^{t_1}, \cdots, p^{t_l}\}$ is that the reduced minors of each order of F generate the unit ideal, where $0 \le t_1 \le \cdots \le t_l$ and $t = t_1 + \cdots + t_l$. Subsequently, Guan, et al.[18] considered the case of $F \in K[x_1, \cdots, x_n]^{l\times l}$ with $\det(F) = p^t$, and established that F is equivalent to its Smith form $\mathrm{diag}\{1, \cdots, 1, \det(F)\}$ if all the $(l-1) \times (l-1)$ minors of F generate the unit ideal, where $l \ge 3$ and $p \in K[x_1]$ is an irreducible polynomial. Recently, Lu, et al.[19] investigated the equivalence problem between a specific class of n-variables polynomial matrices and their Smith forms, where each matrix is characterized by the property that the greatest common divisor of the maximal minors is a univariate polynomial. By employing localization techniques, they fully resolved this issue.

Although significant progress has been made in addressing the equivalence between multivariate polynomial matrices and their Smith forms, the problem remains far from being fully resolved. Therefore, it is still essential to continue investigating the equivalence theory of certain special types of polynomial matrices with respect to their Smith forms. This paper will focus on the following type of bivariate polynomial matrices. Let $F \in K[x, y]^{l\times m}$ with rank r, and $d_r(F) = f(x)(y - g(x))^t$, where $d_r(F)$ is the greatest common divisor of all the $r \times r$

minors of F, and $f, g \in K[x]$. We will investigate the necessary and sufficient condition for the equivalence between F and its Smith form.

The rest of the paper is organized as follows. In Section 2, we first present fundamental concepts related to bivariate polynomial matrices and formulate an equivalence problem that will be addressed in this paper, and then introduce several auxiliary lemmas that will be instrumental in resolving this problem. In Section 3, we first illustrate the main idea of this paper with an example, then address three challenges encountered in the development of the idea, and finally present the necessary and sufficient condition for the equivalence between a bivariate polynomial matrix and its Smith form. In Section 4, we summarize this paper and propose directions for future work.

2 Preliminaries

Let K be a field, $\mathfrak{K}$ be the algebraic closed field containing K, $K[x, y]$ be the polynomial ring in the variables x, y over K, and $K[x, y]^{l\times m}$ be the set of $l \times m$ matrices with entries in $K[x, y]$. Throughout this paper, we assume without loss of generality that $l \le m$.

Let $F \in K[x, y]^{l\times m}$. We use $\text{rank}(F)$ to denote the rank of F. For any integer i, let $I_i(F)$ be the ideal generated by all the $i \times i$ minors of F, and $d_i(F)$ be the greatest common divisor of all the $i \times i$ minors of F. Here, we make the convention that $d_0(F) \equiv 1$ and $d_i(F) \equiv 0$ for $i > \text{rank}(F)$. Given any two strictly increasing sequences of indices $\{i_1, \cdots, i_s\}$ and $\{j_1, \cdots, j_t\}$ with $1 \le i_1 < \cdots < i_s \le l$ and $1 \le j_1 < \cdots < j_t \le m$, denote by $F\begin{pmatrix} i_1 & \cdots & i_s \\ j_1 & \cdots & j_t \end{pmatrix}$ the $s \times t$ submatrix of F formed by its i_1-th to i_s-th rows and j_1-th to j_t-th columns.

2.1 Basic Notions and Equivalence Problem

In this subsection, we first introduce some essential concepts, and then propose the equivalence problem that we will consider. We now present a concept similar to invertible matrices over number fields.

Definition 2.1 Let $U \in K[x, y]^{l\times l}$. Then U is said to be **unimodular** if $\det(U)$ is a nonzero constant in K.

With the help of Definition 2.1, we propose the concept of matrix equivalence over $K[x, y]$.

Definition 2.2 Let $F, Q \in K[x, y]^{l\times m}$. Then F is said to be equivalent to Q if there are two unimodular matrices $U \in K[x, y]^{l\times l}$ and $V \in K[x, y]^{m\times m}$ such that $F = UQV$. For convenience, F being equivalent to Q is denoted by $F \sim Q$.

The Smith form, a well-established concept for matrices over univariate polynomial rings, admits a natural extension to the bivariate case in $K[x, y]$ via an analogous methodology.

Definition 2.3 Let $F \in K[x, y]^{l\times m}$ with rank r, and Φ_i be a polynomial in $K[x, y]$ defined as follows:

$$\Phi_i = \begin{cases} \dfrac{d_i(F)}{d_{i-1}(F)}, & 1 \le i \le r; \\ 0, & r < i \le l. \end{cases}$$

Moreover, Φ_i satisfies the divisibility property $\Phi_1 \mid \Phi_2 \mid \cdots \mid \Phi_r$. Then the **Smith form** of F is given by

$$S = \begin{pmatrix} \mathrm{diag}\{\Phi_1, \cdots, \Phi_r\} & 0_{r\times(m-r)} \\ 0_{(l-r)\times r} & 0_{(l-r)\times(m-r)} \end{pmatrix}.$$

To investigate the problem of equivalence between matrices over $K[x, y]$ and their Smith forms, we further require the following three concepts.

Definition 2.4 (see [20]) Let $F \in K[x, y]^{l\times m}$. Given an integer i, let $a_{i1}, \cdots, a_{i\beta}$ be all the $i \times i$ minors of F, where $1 \le i \le l$ and $\beta = \binom{l}{i}\binom{m}{i}$. Extracting $d_i(F)$ from $a_{i1}, \cdots, a_{i\beta}$ yields

$$a_{ij} = d_i(F) \cdot b_{ij}, \quad j = 1, \cdots, \beta.$$

Then, $b_{i1}, \cdots, b_{i\beta}$ are called the $i \times i$ **reduced minors** of F. For convenience, we use $J_i(F)$ to denote the ideal in $K[x, y]$ generated by $b_{i1}, \cdots, b_{i\beta}$.

The concept of reduced minors plays an important role in this paper, and it is closely related to determining whether a bivariate polynomial matrix is equivalent to its Smith form.

Definition 2.5 (see [21]) Let $\mathcal{I}$ be an ideal in $K[x, y]$. Then we call

$$\mathbb{V}(\mathcal{I}) = \left\{(x_0, y_0) \in \mathfrak{K}^2 \mid h(x_0, y_0) = 0 \text{ for all } h \in \mathcal{I}\right\}$$

the affine variety defined by $\mathcal{I}$.

Li, et al. in [13] proved that $\mathbb{V}(\mathcal{I}) = \emptyset$ if and only if $\mathcal{I} = K[x, y]$.

Definition 2.6 (see [22]) Let $A \in K[x, y]^{l\times m}$ be of full row rank. Then A is said to be zero left prime (**ZLP**) if $I_l(A) = K[x, y]$. Similarly, $A \in K[x, y]^{m\times l}$ can be defined as a zero right prime (**ZRP**) matrix.

We consider a subset of bivariate polynomial matrices as follows:

$$\mathcal{M} := \{F \in K[x, y]^{l\times m} \mid d_r(F) = f(x)(y - g(x))^t, \text{ where } r = \mathrm{rank}(F) \text{ and } f, g \in K[x], t \in \mathbb{N}\}.$$

In the above set, $\mathbb{N}$ represents the set of natural numbers inclusive of 0. When $r = l$ and f is a power of an irreducible polynomial in $K[x]$, Zheng, et al.[23] proved that

$$F \sim Q = \big(\mathrm{diag}\{\underbrace{1, \cdots, 1}_{l-1}, d_l(F)\}, 0_{l\times(m-l)}\big) \text{ if and only if } I_{l-1}(F) = K[x, y].$$

It is evident that Q represents the most particular case among all possible Smith forms of F. In the subsequent analysis, this paper relaxes the constraints imposed in [23], and explores the necessary and sufficient condition for $F \in \mathcal{M}$ to be equivalent to its general Smith form.

Problem 2.7 Let $F \in \mathcal{M}$. What is the necessary and sufficient condition for the equivalence of F and its Smith form?

2.2 Auxiliary Lemmas

In this subsection, we present several lemmas required to address Problem 2.7. We begin by presenting two well-known formulas from linear algebra, both of which are applicable to the polynomial matrix case.

Proposition 2.8 (Laplace expansion formula[24]) *Let $A \in K[x,y]^{l\times l}$. Given any strictly increasing sequence of indices $\{i_1,\cdots,i_k\}$ with $1\le i_1<\cdots<i_k\le l$, we have*

$$\det(A)=\sum_{1\le j_1<\cdots<j_k\le l}\det\Big(A\left(\begin{smallmatrix} i_1 & \cdots & i_k\\ j_1 & \cdots & j_k\end{smallmatrix}\right)\Big)\cdot(-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k}\det\Big(A^{ac}\left(\begin{smallmatrix} i_1 & \cdots & i_k\\ j_1 & \cdots & j_k\end{smallmatrix}\right)\Big),$$

where $A^{ac}\left(\begin{smallmatrix} i_1 & \cdots & i_k\\ j_1 & \cdots & j_k\end{smallmatrix}\right)$ is the $(l-k)\times(l-k)$ submatrix obtained from A by deleting its i_1-th to i_k-th rows and j_1-th to j_k-th columns.

Proposition 2.9 (Binet-Cauchy formula[24]) *Let $A=BC$, where $B\in K[x,y]^{l\times k}$ and $C\in K[x,y]^{k\times m}$. Then an $r\times r$ minor of A is*

$$\det\Big(A\left(\begin{smallmatrix} i_1 & \cdots & i_r\\ j_1 & \cdots & j_r\end{smallmatrix}\right)\Big)=\sum_{1\le s_1<\cdots<s_r\le k}\det\Big(B\left(\begin{smallmatrix} i_1 & \cdots & i_r\\ s_1 & \cdots & s_r\end{smallmatrix}\right)\Big)\cdot\det\Big(C\left(\begin{smallmatrix} s_1 & \cdots & s_r\\ j_1 & \cdots & j_r\end{smallmatrix}\right)\Big),$$

where $1\le r\le \min\{k,l\}$.

Lemma 2.10 *Let $F_1,F_2\in K[x,y]^{l\times m}$. If $F_1\sim F_2$, then $d_i(F_1)=d_i(F_2)$, $I_i(F_1)=I_i(F_2)$ and $J_i(F_1)=J_i(F_2)$, where $i=1,\cdots,l$.*

The proof of Lemma 2.10 follows directly from an application of the Binet-Cauchy formula and is therefore omitted here.

Lemma 2.11 (see [14]) *Let $F,F_1,F_2\in K[x,y]^{l\times l}$ satisfy $F\sim F_1F_2$. If there exists some positive integer k with $k\le l$ such that $J_k(F)=K[x,y]$ and $d_k(F)=d_k(F_1)$, then $J_k(F_1)=J_k(F_2)=K[x,y]$ and $d_k(F_2)=1$.*

Lemma 2.12 (see [19]) *Let $F,F_1,F_2\in K[x,y]^{l\times l}$ satisfy $F=F_1F_2$ and* $\gcd(\det(F_1),\det(F_2))=1$. *Then*

1) *$d_i(F)=d_i(F_1)\cdot d_i(F_2)$, where $i=1,\cdots,l$;*

2) *If $J_i(F)=K[x,y]$, then $J_i(F_1)=J_i(F_2)=K[x,y]$, where $i=1,\cdots,l$.*

In 1976, Quillen[25] and Suslin[26] independently resolved the renowned Serre Conjecture, and thereby established a connection between ZLP matrices and unimodular matrices, a result now known as the Quillen-Suslin theorem. Specifically, a ZLP matrix can be embedded into a unimodular matrix. Based on the Quillen-Suslin theorem, the following conclusion can be easily derived.

Corollary 2.13 *Let $A\in K[x,y]^{l\times m}$ be a ZLP matrix. Then there exists a unimodular matrix $U\in K[x,y]^{m\times m}$ such that $AU=(\boldsymbol{I}_l,0_{l\times(m-l)})$, where $\boldsymbol{I}_l$ is the $l\times l$ identity matrix.*

By leveraging the Quillen-Suslin theorem, Wang and Feng[27] resolved the Lin-Bose Conjecture proposed in [28], and further extended the result to the case of rank-deficient matrices.

Lemma 2.14 *Let $F \in K[x,y]^{l\times m}$ with $\operatorname{rank}(F) = r$, and $J_r(F) = K[x,y]$, where $1 \le r \le l$. Then there exist $G_1 \in K[x,y]^{l\times r}$ and $F_1 \in K[x,y]^{r\times m}$ such that $F = G_1F_1$ with F_1 being a ZLP matrix.*

Theorem 2.15 (Primitive factorization theorem[29]) *Let $F \in K[x,y]^{l\times m}$ be of full row rank, and $h \in K[x]$ be a divisor of $d_l(F)$. Then there exist $G_1 \in K[x,y]^{l\times l}$ and $F_1 \in K[x,y]^{l\times m}$ such that $F = G_1F_1$ with $\det(G_1) = h$.*

The key idea of the primitive factorization theorem is to establish a homomorphism from $K[x,y]$ to a Euclidean integral domain, and then use Gaussian elimination method to factorize F. This conclusion will help us in constructing the process of matrix equivalence in this paper.

Lemma 2.16 (see [19]) *Let $F \in K[x,y]^{l\times m}$ with rank r, and $d_r(F) = f(x)$ be a univariate polynomial. Then F is equivalent to its Smith form if and only if $J_i(F) = K[x,y]$ with $i = 1,\cdots,r$.*

As observed from Problem 2.7, the polynomial matrices investigated in this paper are different from those in Lemma 2.16. Although it is only a change from $d_r(F)$ being $f(x)$ to $f(x)(y-g(x))^t$, it has brought many difficulties to solving Problem 2.7. These challenges will be elaborated in the next section.

3 Equivalence Theory

This section is devoted to presenting the detailed solution procedure for Problem 2.7, and thus contains a number of technical proof processes. To enhance readability, Table 1 summarizing recurrent notations along with their corresponding interpretations is provided.

Table 1 Notations

No.	Notation	Description
1	p	the bivariate polynomial $y - g(x)$
2	$d_i(F)$	the greatest common divisor of all the $i\times i$ minors of F
3	$I_i(F)$	the ideal generated by all the $i\times i$ minors of F
4	$J_i(F)$	the ideal generated by all the $i\times i$ reduced minors of F
5	$F \sim Q$	F is equivalent to Q
6	$\langle h_1,\cdots,h_l\rangle$	the ideal generated by $h_1,\cdots,h_l$
7	$\vec{a} \in A \setminus A_1$	$\vec{a}$ belongs to the set formed by the rows of A after removing A_1

3.1 Main Idea and Challenges

To address Problem 2.7, we will use an example to illustrate the main idea of the solution procedure and the challenges encountered therein.

Example 3.1 Let $F \in K[x,y]^{4\times 4}$ with $\det(F) = fp^6$, where $f \in K[x]$. Assume that the Smith form of F is $S = \operatorname{diag}\{f_1, f_2p, f_3p^2, f_4p^3\}$, where $f_1 \mid f_2 \mid f_3 \mid f_4$ and $f = f_1f_2f_3f_4$. In addition, $J_i(F) = K[x,y]$ for $i = 1,\cdots,4$.

According to the primitive factorization theorem, there are two matrices $G_1, F_1 \in K[x,y]^{4\times 4}$ such that $F = G_1F_1$ with $\det(G_1) = f$. It follows from the Binet-Cauchy formula that $\det(F_1) =$

p^6. This implies that $\gcd(\det(G_1), \det(F_1)) = 1$. Based on the first result of Lemma 2.12, we have $d_i(F) = d_i(G_1)d_i(F_1)$, where $i = 1, \cdots, 4$. It follows that $d_i(G_1) = f_1 \cdots f_i$ for $i = 1, \cdots, 4$, and $d_1(F_1) = 1, d_2(F_1) = p, d_3(F_1) = p^3, d_4(F_1) = p^6$.

Under the assumption that $J_i(F) = K[x, y]$ for $i = 1, \cdots, 4$, we can utilize the second result of Lemmas 2.12 and 2.16 to obtain the conclusion that

$$G_1 \sim S_{G_1} = \mathrm{diag}\{f_1, f_2, f_3, f_4\}.$$

There exist two unimodular matrices $U_1, V_1 \in K[x, y]^{4\times 4}$ such that $G_1 = U_1 S_{G_1} V_1$. Then, we get $F = U_1 S_{G_1} V_1 F_1$. Let $F_2 = V_1 F_1$. By the fact that V_1 is unimodular, $d_i(F_2) = d_i(F_1)$ and $J_i(F_2) = J_i(F_1)$, where $i = 1, \cdots, 4$.

The first thing we want to do is to prove that there exist a unimodular matrix $U_2 \in K[x]^{4\times 4}$ and a polynomial matrix $F_3 \in K[x, y]^{4\times 4}$ such that $F_2 = U_2 \cdot \mathrm{diag}\{1, p, p, p\} \cdot F_3$. This is the first challenge. By overcoming this challenge, we can obtain

$$F \sim \mathrm{diag}\{f_1, f_2, f_3, f_4\} \cdot U_2 \cdot \mathrm{diag}\{1, p, p, p\} \cdot F_3.$$

The second challenge is to prove that

$$\mathrm{diag}\{f_1, f_2, f_3, f_4\} \cdot U_2 \cdot \mathrm{diag}\{1, p, p, p\} \sim \mathrm{diag}\{f_1, f_2 p, f_3 p, f_4 p\}.$$

After solving this challenge, we can get

$$F \sim \mathrm{diag}\{f_1, f_2 p, f_3 p, f_4 p\} \cdot F_4,$$

where $F_4 \in K[x, y]^{4\times 4}$.

Finally, the third challenge is to prove that there exist a unimodular matrix $U_3 \in K[x]^{4\times 4}$ and a polynomial matrix $F_5 \in K[x, y]^{4\times 4}$ such that $F_4 = U_3 \cdot \mathrm{diag}\{1, 1, p, p\} \cdot F_5$. If we can prove, then we have

$$F \sim \mathrm{diag}\{f_1, f_2 p, f_3 p, f_4 p\} \cdot U_3 \cdot \mathrm{diag}\{1, 1, p, p\} \cdot F_5.$$

By repeatedly applying the above calculation process, we can deduce the result:

$$F \sim S = \mathrm{diag}\{f_1, f_2 p, f_3 p^2, f_4 p^3\}.$$

In Example 3.1, the first and the third challenges are closely related. In the subsequent subsection, we will address these two challenges first. The second challenge is the most crucial part in solving Problem 2.7, and we will tackle it at the end.

3.2 Solving the Challenges

Leveraging the particular structure of p, we construct the following homomorphism

$$\begin{aligned} \phi_p : \quad & K[x, y] \longrightarrow K[x], \\ & h(x, y) \longrightarrow h(x, g(x)). \end{aligned}$$

This homomorphism can extend canonically to the homomorphism $\phi_p : K[x, y]^{l\times m} \to K[x]^{l\times m}$ by applying ϕ_p entry-wise.

Lemma 3.2 *Let $A \in K[x,y]^{l\times m}$. If $\mathrm{rank}(\phi_p(A)) = k$, then there exist $A_1 \in K[x,y]^{l\times m}$ and a unimodular matrix $U \in K[x]^{l\times l}$ such that*

$$A = U \cdot \mathrm{diag}\{1, \cdots, 1, \underbrace{p, \cdots, p}_{l-k}\} \cdot A_1.$$

Proof Note that $K[x]$ is a Euclidean domain and $\phi_p(A) \in K[x]^{l\times m}$. There is a unimodular matrix $V \in K[x]^{l\times l}$ such that $V\phi_p(A) = A_0$, where $A_0 \in K[x]^{l\times m}$ is a upper triangular matrix and the last $l-k$ rows of A_0 are zero vectors. Let $A' = VA$. It follows from ϕ_p being a homomorphism that

$$\phi_p(A') = \phi_p(V)\phi_p(A) = V\phi_p(A) = A_0.$$

This implies that all elements of the last $l-k$ rows of A' are divisible by p, i.e.,

$$A' = \mathrm{diag}\{1, \cdots, 1, \underbrace{p, \cdots, p}_{l-k}\} \cdot A_1,$$

where $A_1 \in K[x,y]^{l\times m}$. Let $U = V^{-1}$, we have $A = U \cdot \mathrm{diag}\{1, \cdots, 1, p, \cdots, p\} \cdot A_1$. The proof is completed. ∎

Lemma 3.3 *Let $A \in K[x,y]^{l\times m}$, and $A_1 \in K[x,y]^{k\times m}$ be a submatrix of A such that $p \nmid d_k(A_1)$. If for any row vector $\vec{a} \in A \setminus A_1$, the matrix $A_2 = \begin{pmatrix} A_1 \\ \vec{a} \end{pmatrix}$ satisfies $p \mid d_{k+1}(A_2)$, then $\mathrm{rank}(\phi_p(A)) = k$.*

Proof Without loss of generality, assume that $A_1 = (\vec{a}_1^{\mathrm{T}}, \cdots, \vec{a}_k^{\mathrm{T}})^{\mathrm{T}}$, where $\vec{a}_i \in K[x,y]^{1\times m}$ is the i-th row of A, $i = 1, \cdots, k$. Since $p \nmid d_k(A_1)$, there exists at least a $k\times k$ minor $b \in K[x,y]$ of A_1 such that $p \nmid b$. Then $\phi_p(b) \neq 0$, and it follows that $\mathrm{rank}(\phi_p(A_1)) = k$. Since A_1 is a submatrix of A, $\mathrm{rank}(\phi_p(A)) \geq k$. Similarly, it follows from $p \mid d_{k+1}(A_2)$ that $\mathrm{rank}(\phi_p(A_2)) \leq k$. Since $A_2 = \begin{pmatrix} A_1 \\ \vec{a} \end{pmatrix}$, $\phi_p(\vec{a}_1), \cdots, \phi_p(\vec{a}_k), \phi_p(\vec{a})$ are $K[x]$-linearly dependent. Then $\forall \vec{a} \in A \setminus A_1$, there are $h_1, \cdots, h_k \in K(x)$ such that

$$\phi_p(\vec{a}) = h_1\phi_p(\vec{a}_1) + \cdots + h_k\phi_p(\vec{a}_k), \tag{1}$$

where $K(x)$ is the rational fraction field of $K[x]$. For any given $k+1$ row vectors $\vec{a}_{t_1}, \cdots, \vec{a}_{t_{k+1}}$ of A, by Equation (1) we have

$$\begin{pmatrix} \phi_p(\vec{a}_{t_1}) \\ \vdots \\ \phi_p(\vec{a}_{t_k}) \\ \phi_p(\vec{a}_{t_{k+1}}) \end{pmatrix} = \begin{pmatrix} h_{t_1 1} & \cdots & h_{t_1 k} \\ \vdots & \ddots & \vdots \\ h_{t_k 1} & \cdots & h_{t_k k} \\ h_{t_{k+1} 1} & \cdots & h_{t_{k+1} k} \end{pmatrix} \begin{pmatrix} \phi_p(\vec{a}_1) \\ \vdots \\ \phi_p(\vec{a}_k) \end{pmatrix}, \tag{2}$$

where $h_{t_1 1}, \cdots, h_{t_{k+1} k} \in K(x)$. We write Equation (2) as $B = H \cdot \phi_p(A_1)$. It follows from $\mathrm{rank}(B) \leq \min\{\mathrm{rank}(H), \mathrm{rank}(\phi_p(A_1))\}$ that $\mathrm{rank}(B) \leq k$ over $K(x)$. Then there is a nonzero vector $\vec{w} \in K(x)^{1\times(k+1)}$ such that $\vec{w}B = \vec{0}$. Multiplying both sides of the above equation by the least common multiple of all the denominators of entries in $\vec{w}$, we get

$$w_1\phi_p(\vec{a}_{t_1}) + \cdots + w_k\phi_p(\vec{a}_{t_k}) + w_{k+1}\phi_p(\vec{a}_{t_{k+1}}) = \vec{0}, \tag{3}$$

where $w_1, \cdots, w_k, w_{k+1} \in K[x]$ with at least one $w_j \neq 0$. It follows from Equation (3) that any $k+1$ row vectors of $\phi_p(A)$ are $K[x]$-linearly dependent. Therefore, $\operatorname{rank}(\phi_p(A)) = k$. ∎

Lemma 3.4 *Let $A \in K[x,y]^{l\times l}$, and the Smith form of A be $S_A = \operatorname{diag}\{f_1p^{s_1}, \cdots, f_lp^{s_l}\}$, where $f_1, \cdots, f_l \in K[x]$ satisfy that $f_1 \mid f_2 \mid \cdots \mid f_l$, and $0 \le s_1 \le \cdots \le s_l$. Assume that there exists a matrix $B \in K[x,y]^{l\times l}$ such that*

$$A \sim \operatorname{diag}\{f_1p^{s_1}, \cdots, f_kp^{s_k}, f_{k+1}p^{s}, \cdots, f_lp^{s}\} \cdot B,$$

where $s_k \le s < s_{k+1}$. If $J_k(A) = K[x,y]$, then $\operatorname{rank}(\phi_p(B)) = k$.

Proof Let $\Lambda = \operatorname{diag}\{f_1p^{s_1}, \cdots, f_kp^{s_k}, f_{k+1}p^{s}, \cdots, f_lp^{s}\}$. It is easy to check that $d_k(A) = d_k(\Lambda) = f_1\cdots f_kp^{s_1+\cdots+s_k}$. According to Lemma 2.11, it follows from $A \sim \Lambda B$ and $J_k(A) = K[x,y]$ that $J_k(B) = K[x,y]$ and $d_k(B) = 1$. This implies that $I_k(B) = J_k(B) = K[x,y]$. It follows that there exists at least one submatrix $B_1 \in K[x,y]^{k\times l}$ of B such that $p \nmid d_k(B_1)$. Next, we prove that $p \mid d_{k+1}\left(\begin{pmatrix} B_1 \\ \vec{b} \end{pmatrix}\right)$ for each vector $\vec{b} \in B \setminus B_1$. Next, we divide the proof into three parts.

First: $s_1 = \cdots = s_k = s$.

Let $C = \Lambda B$ and B_1 be formed by the i_1-th, i_2-th, $\cdots$, i_k-th rows of B, where $1 \le i_1 < i_2 < \cdots < i_k \le l$. Since $A \sim \Lambda B$, $d_{k+1}(C) = d_{k+1}(A) = f_1\cdots f_kf_{k+1}p^{ks+s_{k+1}}$. Let $\vec{b}_j$ be the j-th row of B and $D_j = \begin{pmatrix} B_1 \\ \vec{b}_j \end{pmatrix}$, where $\vec{b}_j \in B \setminus B_1$. It follows from $d_{k+1}(C) \mid f_{i_1}\cdots f_{i_k}f_jp^{(k+1)s}d_{k+1}(D_j)$ and $s < s_{k+1}$ that $p \mid d_{k+1}(D_j)$. According to Lemma 3.3, we get $\operatorname{rank}(\phi_p(B)) = k$.

Second: $s_1 < s_2 < \cdots < s_k < s$.

In this case, let $C = \Lambda B$ and $B_1' \in K[x,y]^{k\times l}$ be the submatrix of B formed by its first k rows. We assert that $p \nmid d_k(B_1')$. Otherwise, all the $k \times k$ minors of C have a common divisor $p^{s_1+\cdots+s_k+1}$. This contradicts the fact that $d_k(C) = f_1\cdots f_kp^{s_1+\cdots+s_k}$. Let $B_1 = B_1'$, $\vec{b}_j$ be the j-th row of B and $D_j = \begin{pmatrix} B_1 \\ \vec{b}_j \end{pmatrix}$, where $j = k+1, \cdots, l$. Then for each j, we have $d_{k+1}(C) \mid f_1\cdots f_kf_jp^{s_1+\cdots+s_k+s}d_{k+1}(D_j)$. By the fact that $d_{k+1}(C) = f_1\cdots f_kf_{k+1}p^{s_1+\cdots+s_k+s_{k+1}}$ and $s < s_{k+1}$, we have $p \mid d_{k+1}(D_j)$, where $j = k+1, \cdots, l$. According to Lemma 3.3, we obtain $\operatorname{rank}(\phi_p(B)) = k$.

Third: The remaining cases except the first and second. The same conclusion can be derived through the above method.

Therefore, the proof is completed. ∎

Remark 3.5 It follows from $d_i(F_2) = d_i(F_1)$ in Example 3.1 that $d_1(F_2) = 1$ and $d_2(F_2) = p$. It is easy to check that $\operatorname{rank}(\phi_p(F_2)) = 1$. Then Lemma 3.2 can be used to resolve the first challenge. When $F \sim \operatorname{diag}\{f_1, f_2p, f_3p, f_4p\} \cdot F_4$, by Lemma 3.4 we have $\operatorname{rank}(\phi_p(F_4)) = 2$. Using Lemma 3.2 again, we can solve the third challenge.

Lemma 3.6 *Let*

$$A = \operatorname{diag}\{h_1, \cdots, h_k, \underbrace{h, \cdots, h}_{l-k}\} \cdot \begin{pmatrix} \boldsymbol{I}_k & \\ & B \end{pmatrix},$$

where $h_1, \cdots, h_k, h \in K[x,y]$ satisfy $h_1 \mid h_2 \mid \cdots \mid h_k \mid h$, and $B \in K[x,y]^{(l-k)\times(l-k)}$. Then for $j = k+1, \cdots, l$, we have

1) $d_j(A) = h_1 \cdots h_k h^{j-k} \cdot d_{j-k}(B)$;

2) *If* $d_j(A) = h_1 \cdots h_k h^{j-k}$, *then* $J_j(A) = I_{j-k}(B)$.

Proof For any two given strictly increasing index sets $\{s_1, \cdots, s_{j-k}\}$ and $\{t_1, \cdots, t_{j-k}\}$ that satisfy $1 \leq s_1 < \cdots < s_{j-k} \leq l-k$ and $1 \leq t_1 < \cdots < t_{j-k} \leq l-k$, the following determinant

$$\begin{aligned} &\det\left(A\begin{pmatrix}1 & \cdots & k & (k+s_1) & \cdots & (k+s_{j-k})\\ 1 & \cdots & k & (k+t_1) & \cdots & (k+t_{j-k})\end{pmatrix}\right)\\ =&h_1\cdots h_k h^{j-k}\det\left(B\begin{pmatrix}s_1 & \cdots & s_{j-k}\\ t_1 & \cdots & t_{j-k}\end{pmatrix}\right) \end{aligned} \tag{4}$$

is a $j \times j$ minor of A. Let $\alpha_1, \cdots, \alpha_N \in K[x, y]$ be all the $(j-k) \times (j-k)$ minors of B, where $N = \binom{l-k}{j-k}^2$. It follows from Equation (4) that

$$h_1 \cdots h_k h^{j-k}\alpha_1, \cdots, h_1 \cdots h_k h^{j-k}\alpha_N \tag{5}$$

are some of all the $j \times j$ minors of A. Obviously,

$$\gcd(h_1 \cdots h_k h^{j-k}\alpha_1, \cdots, h_1 \cdots h_k h^{j-k}\alpha_N) = h_1 \cdots h_k h^{j-k} \cdot d_{j-k}(B).$$

Let $\beta_1, \cdots, \beta_\eta \in K[x, y]$ be all the $j \times j$ minors of A excluding the elements in Equation (5), where $\eta = \binom{l}{j}^2 - N$. Without loss of generality, assume that

$$\beta_1 = \det\left(A\begin{pmatrix}1 & \cdots & k-1 & k+1 & \cdots & j+1\\ 1 & \cdots & k-1 & k+1 & \cdots & j+1\end{pmatrix}\right).$$

Then we have

$$\beta_1 = h_1 \cdots h_{k-1} h^{j-k+1}\det\left(B\begin{pmatrix}1 & \cdots & j-k+1\\ 1 & \cdots & j-k+1\end{pmatrix}\right). \tag{6}$$

Since $\det\left(B\left(\begin{smallmatrix}1 & \cdots & j-k+1\\ 1 & \cdots & j-k+1\end{smallmatrix}\right)\right)$ is a $(j-k+1) \times (j-k+1)$ minor of B, it follows from the Laplace expansion formula that

$$d_{j-k}(B) \mid \det\left(B\begin{pmatrix}1 & \cdots & j-k+1\\ 1 & \cdots & j-k+1\end{pmatrix}\right).$$

This implies that

$$h_1 \cdots h_k h^{j-k} \cdot d_{j-k}(B) \mid \beta_1.$$

By the same method, we can deduce that

$$h_1 \cdots h_k h^{j-k} \cdot d_{j-k}(B) \mid \beta_i, \text{ where } i = 1, \cdots, \eta.$$

Hence, it follows that

$$\begin{aligned} d_j(A) &= \gcd\left(h_1\cdots h_k h^{j-k}\alpha_1,\cdots,h_1\cdots h_k h^{j-k}\alpha_N,\beta_1,\cdots,\beta_\eta\right)\\ &= \gcd\left(\gcd(h_1\cdots h_k h^{j-k}\alpha_1,\cdots,h_1\cdots h_k h^{j-k}\alpha_N),\beta_1,\cdots,\beta_\eta\right)\\ &= \gcd\left(h_1\cdots h_k h^{j-k}\cdot d_{j-k}(B),\beta_1,\cdots,\beta_\eta\right)\\ &= h_1\cdots h_k h^{j-k}\cdot d_{j-k}(B). \end{aligned}$$

If $d_j(A) = h_1\cdots h_k h^{j-k}$, then

$$J_j(A) = \left\langle \alpha_1,\cdots,\alpha_N,\frac{\beta_1}{h_1\cdots h_k h^{j-k}},\cdots,\frac{\beta_\eta}{h_1\cdots h_k h^{j-k}}\right\rangle. \tag{7}$$

It follows from Equation (6) that

$$\frac{\beta_1}{h_1\cdots h_k h^{j-k}} = \frac{h}{h_k}\det\left(B\begin{pmatrix}1 & \cdots & j-k+1\\ 1 & \cdots & j-k+1\end{pmatrix}\right).$$

According to the Laplace expansion formula, $\det\left(B\left(\begin{smallmatrix}1 & \cdots & j-k+1\\ 1 & \cdots & j-k+1\end{smallmatrix}\right)\right)$ can be expressed as a linear combination of $\alpha_1,\cdots,\alpha_N$ over $K[x,y]$. This implies that

$$\frac{\beta_1}{h_1\cdots h_k h^{j-k}} \in \langle \alpha_1,\cdots,\alpha_N\rangle.$$

Adopting the same reasoning, it follows that

$$\frac{\beta_i}{h_1\cdots h_k h^{j-k}} \in \langle \alpha_1,\cdots,\alpha_N\rangle, \quad \text{where } i = 1,\cdots,\eta. \tag{8}$$

Combining Equations (7) and (8), we get

$$J_j(A) = \langle \alpha_1,\cdots,\alpha_N\rangle = I_{j-k}(B).$$

The proof is completed. ∎

Lemma 3.7 *Let $W = \Lambda_1 U \Lambda_2$, where $\Lambda_1 = \mathrm{diag}\{h_1,\cdots,h_l\}$ with $h_1,\cdots,h_l \in K[x,y]$ satisfying $h_1 \mid h_2 \mid \cdots \mid h_l$, $U \in K[x]^{l\times l}$ is a unimodular matrix, and $\Lambda_2 = \mathrm{diag}\{p^{t_1},\cdots,p^{t_l}\}$ with the exponents satisfying $0 \le t_1 \le \cdots \le t_l$. Then $d_k(W) = d_k(\Lambda_1)\cdot d_k(\Lambda_2)$, where $k = 1,\cdots,l$.*

Proof Let $U = (u_{ij})_{l\times l}$, where $u_{ij} \in K[x]$ for $1 \le i,j \le l$. Since $W = \Lambda_1 U \Lambda_2$, we get

$$W = \begin{pmatrix} h_1p^{t_1}u_{11} & h_1p^{t_2}u_{12} & \cdots & h_1p^{t_l}u_{1l}\\ h_2p^{t_1}u_{21} & h_2p^{t_2}u_{22} & \cdots & h_2p^{t_l}u_{2l}\\ \vdots & \vdots & \ddots & \vdots\\ h_lp^{t_1}u_{l1} & h_lp^{t_2}u_{l2} & \cdots & h_lp^{t_l}u_{ll} \end{pmatrix}.$$

For any given integer k with $1 \le k \le l$, let W_k be the matrix formed by the first k rows of W. Then all the $k\times k$ minors of W_k are $h_1\cdots h_k p^{t_1+\cdots+t_k}\alpha_1, h_1\cdots h_k p^{\theta_2}\alpha_2,\cdots,h_1\cdots h_k p^{\theta_N}\alpha_N$, where $N = \binom{l}{k}$, and $\theta_i = t_{i_1} + \cdots + t_{i_k}$, the indices $\{i_1, \cdots, i_k\}$ is a strictly increasing sequence with $1 \leq i_1 < \cdots < i_k \leq l$, $i = 2, \cdots, N$. Obviously, $\theta_i \geq t_1 + \cdots + t_k$ for $i = 2, \cdots, N$, $\alpha_1, \cdots, \alpha_N \in K[x]$ are all the $k \times k$ minors of the matrix U_k formed by the fist k rows of U. Since U is unimodular, it follows from the Laplace expansion formula that $\alpha_1, \cdots, \alpha_N$ generate the unit ideal $K[x]$. Let $\theta'_i = \theta_i - (\sum_{j=1}^{k} t_j)$, where $i = 2, \cdots, N$. Assume that

$$a = \gcd(\alpha_1, p^{\theta'_2}\alpha_2, \cdots, p^{\theta'_N}\alpha_N),$$

where $a \in K[x, y]$. It follows from $a \mid \alpha_1$ that $a \in K[x]$. As $p = y - g(x)$ is irreducible, we have $a \mid \alpha_i$ for $i = 2, \cdots, N$. Since $\langle \alpha_1, \cdots, \alpha_N \rangle = 1$, we get $a = 1$. It follows that

$$d_k(W_k) = h_1 \cdots h_k p^{t_1 + \cdots + t_k}.$$

On the one hand, it follows from $h_1 \mid h_2 \mid \cdots \mid h_l$ and $t_1 \leq t_2 \leq \cdots \leq t_l$ that $h_1 \cdots h_k p^{t_1+\cdots+t_k}$ divides every $k \times k$ minor of W. This implies that $d_k(W_k) \mid d_k(W)$. On the other hand, since all the $k \times k$ minors of W_k are part of those of W, we have $d_k(W) \mid d_k(W_k)$. Therefore, $d_k(W) = d_k(W_k)$. It follows from $d_k(\Lambda_1) = h_1 \cdots h_k$ and $d_k(\Lambda_2) = p^{t_1+\cdots+t_k}$ that $d_k(W) = d_k(\Lambda_1) \cdot d_k(\Lambda_2)$. The proof is completed. ∎

Remark 3.8 Lemma 3.7 is different from the first result of Lemma 2.12. This is because $\gcd(\det(\Lambda_1), \det(\Lambda_2))$ may be a nontrivial polynomial in $K[x, y]$.

Lemma 3.9 *Let*

$$A = \mathrm{diag}\{f_1 p^{s_1}, \cdots, f_k p^{s_k}, f_{k+1} p^s, \cdots, f_l p^s\} \cdot U \cdot \mathrm{diag}\{\underbrace{1, \cdots, 1}_{k}, p, \cdots, p\},$$

where $f_1, \cdots, f_l \in K[x]$ satisfy $f_1 \mid f_2 \mid \cdots \mid f_l$, $0 \leq s_1 \leq \cdots \leq s_k \leq s$, and $U \in K[x]^{l \times l}$ is a unimodular matrix. If $J_i(A) = K[x, y]$ for $i = 1, \cdots, k$, then

$$A \sim \mathrm{diag}\{f_1 p^{s_1}, \cdots, f_k p^{s_k}, f_{k+1} p^{s+1}, \cdots, f_l p^{s+1}\}.$$

Proof Under the premise that $s_1 \leq \cdots \leq s_k \leq s$, we partition the proof into three parts based on their ordering.

First: $s_1 = \cdots = s_k = s$.

The above problem is equivalent to proving that

$$\mathrm{diag}\{f_1, \cdots, f_k, f_{k+1}, \cdots, f_l\} \cdot U \cdot \mathrm{diag}\{1, \cdots, 1, p, \cdots, p\} \sim \mathrm{diag}\{f_1, \cdots, f_k, f_{k+1}p, \cdots, f_l p\}.$$

Let $A_1 = \mathrm{diag}\{f_1, \cdots, f_k, f_{k+1}, \cdots, f_l\} \cdot U \cdot \mathrm{diag}\{1, \cdots, 1, p, \cdots, p\}$. Then $A = p^s A_1$. Since $J_i(A) = K[x, y]$ with $i = 1, \cdots, k$, we have $J_i(A_1) = K[x, y]$, where $i = 1, \cdots, k$. Based on Lemma 3.7, we have

$$d_i(A_1) = f_1 \cdots f_i \text{ for } 1 \leq i \leq k, \text{ and } d_j(A_1) = f_1 \cdots f_j p^{j-k} \text{ for } k+1 \leq j \leq l.$$

Let $U = (u_{ij})_{l\times l}$, where $u_{ij} \in K[x]$. Then

$$A_1 = \begin{pmatrix} f_1u_{11} & \cdots & f_1u_{1k} & f_1pu_{1(k+1)} & \cdots & f_1pu_{1l} \\ f_2u_{21} & \cdots & f_2u_{2k} & f_2pu_{2(k+1)} & \cdots & f_2pu_{2l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_lu_{l1} & \cdots & f_lu_{lk} & f_lpu_{l(k+1)} & \cdots & f_lpu_{ll} \end{pmatrix}.$$

Let $B_1 \in K[x]^{l\times k}$ be the matrix formed by the first k columns of A_1. Without loss of generality, assume that the Smith form of B_1 is

$$S_{B_1} = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_k \\ & & & \\ & & & \end{pmatrix},$$

where $b_i \in K[x]$ for $i = 1, \cdots, k$ and $b_1 \mid b_2 \mid \cdots \mid b_k$. Since B_1 is a univariate polynomial matrix in $K[x]$, there exist two unimodular matrices $U_{B_1} \in K[x]^{l\times l}$ and $V_{B_1} \in K[x]^{k\times k}$ such that $U_{B_1}B_1V_{B_1} = S_{B_1}$. Let

$$A_2 = U_{B_1} \cdot A_1 \cdot \begin{pmatrix} V_{B_1} & \\ & \boldsymbol{I}_{l-k} \end{pmatrix},$$

where $\boldsymbol{I}_{l-k}$ is the $(l-k)\times(l-k)$ identity matrix. Then $A_1 \sim A_2$ by the fact that U_{B_1} and $\begin{pmatrix} V_{B_1} & \\ & \boldsymbol{I}_{l-k} \end{pmatrix}$ are two unimodular matrices. By Lemma 2.10, we have

$$J_i(A_2) = K[x,y] \text{ and } d_i(A_2) = f_1\cdots f_i \text{ for } 1 \le i \le k,\ d_j(A_2) = f_1\cdots f_jp^{j-k} \text{ for } k+1 \le j \le l.$$

In addition,

$$A_2 = \begin{pmatrix} b_1 & & & & pv_{1(k+1)} & \cdots & pv_{1l} \\ & b_2 & & & pv_{2(k+1)} & \cdots & pv_{2l} \\ & & \ddots & & \vdots & \ddots & \vdots \\ & & & b_k & pv_{k(k+1)} & \cdots & pv_{kl} \\ & & & & \vdots & \ddots & \vdots \\ & & & & pv_{l(k+1)} & \cdots & pv_{ll} \end{pmatrix},$$

where $v_{ij} \in K[x]$ for $1 \le i \le l$ and $k+1 \le j \le l$. It follows from $d_1(A_2) = f_1$ that

$$A_2 = f_1 \cdot \begin{pmatrix} b_1' & & & & pv'_{1(k+1)} & \cdots & pv'_{1l} \\ & b_2' & & & pv'_{2(k+1)} & \cdots & pv'_{2l} \\ & & \ddots & & \vdots & \ddots & \vdots \\ & & & b_k' & pv'_{k(k+1)} & \cdots & pv'_{kl} \\ & & & & \vdots & \ddots & \vdots \\ & & & & pv'_{l(k+1)} & \cdots & pv'_{ll} \end{pmatrix} = f_1 \cdot A_3,$$

where $A_3 \in K[x,y]^{l\times l}$. Moreover, $J_1(A_2) = I_1(A_3) = K[x,y]$. Since $b_i \mid b_j$ for $1 \le i < j \le k$, we have $b_i' \mid b_j'$ for $1 \le i < j \le k$. We assert that b_1' is a nonzero constant in K. Otherwise, if $b_1' = 0$, then $d_l(A_2) = f_1^l d_l(A_3) = 0$. This contradicts the fact that $d_l(A_2) = f_1 \cdots f_l p^{l-k}$. If b_1' is a nontrivial polynomial in $K[x]$, then there exists a point $x_0 \in \mathfrak{K}$ such that $b_1'(x_0) = 0$. It follows from $b_1' \mid b_j'$ that $b_j'(x_0) = 0$, where $j = 2, \cdots, k$. Letting $y_0 = g(x_0)$, we get $p(x_0, y_0) = y_0 - g(x_0) = 0$. This implies that $(x_0, y_0) \in \mathbb{V}(I_1(A_3))$. This contradicts the fact that $I_1(A_3) = K[x,y]$. Without loss of generality, let $b_1' = 1$. Then it is easy to check that

$$A_3 \sim \begin{pmatrix} 1 & & & & & & \\ & b_2' & & & pv'_{2(k+1)} & \cdots & pv'_{2l} \\ & & \ddots & & \vdots & \ddots & \vdots \\ & & & b_k' & pv'_{k(k+1)} & \cdots & pv'_{kl} \\ & & & & \vdots & \ddots & \vdots \\ & & & & pv'_{l(k+1)} & \cdots & pv'_{ll} \end{pmatrix} = \begin{pmatrix} 1 & \\ & A_4 \end{pmatrix},$$

where $A_4 \in K[x,y]^{(l-1)\times(l-1)}$. Based on Lemma 3.6, we obtain

$$d_2\left(\begin{pmatrix} 1 & \\ & A_4 \end{pmatrix}\right) = d_1(A_4).$$

Since $A_2 = f_1 A_3$ and $d_2(A_2) = f_1 f_2$, we get $d_2(A_3) = \frac{f_2}{f_1}$. It follows from $A_3 \sim \begin{pmatrix} 1 & \\ & A_4 \end{pmatrix}$ that

$$d_1(A_4) = d_2(A_3) = \frac{f_2}{f_1}.$$

Then,

$$A_4 = \frac{f_2}{f_1} \cdot \begin{pmatrix} b_2'' & & & pv''_{2(k+1)} & \cdots & pv''_{2l} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & b_k'' & pv''_{k(k+1)} & \cdots & pv''_{kl} \\ & & & \vdots & \ddots & \vdots \\ & & & pv''_{l(k+1)} & \cdots & pv''_{ll} \end{pmatrix} = \frac{f_2}{f_1} \cdot A_5,$$

where $A_5 \in K[x,y]^{(l-1)\times(l-1)}$. Moreover, $b''_i \mid b''_j$ for $2 \le i < j \le k$. Then we assert that b''_2 is a nonzero constant in K. Otherwise, if $b''_2 = 0$, then $d_l(A_3) = d_l\left(\begin{pmatrix} 1 & \\ & A_4 \end{pmatrix}\right) = 0$, which leads to a contradiction. If b''_2 is a nontrivial polynomial in $K[x]$, then there exists a point $x_1 \in \mathfrak{K}$ such that $b''_2(x_1) = 0$. It follows from $b''_2 \mid b''_j$ that $b''_j(x_1) = 0$, where $j = 3, \cdots, k$. This implies that $(x_1, g(x_1)) \in \mathbb{V}(I_1(A_5))$. Let

$$A_6 = f_1 \cdot \begin{pmatrix} 1 & \\ & \frac{f_2}{f_1} \cdot A_5 \end{pmatrix} = \mathrm{diag}\{f_1, f_2, \cdots, f_2\} \cdot \begin{pmatrix} 1 & \\ & A_5 \end{pmatrix}.$$

Then $A_2 \sim A_6$. It follows that $d_2(A_6) = f_1 f_2$ and $J_2(A_6) = K[x,y]$. Furthermore, by Lemma 3.6 we get $J_2(A_6) = I_1(A_5)$. This implies that $(x_1, g(x_1)) \in \mathbb{V}(J_2(A_6))$, which leads to a contradiction. Without loss of generality, assume that $b''_2 = 1$. Then we have

$$A_5 \sim \begin{pmatrix} 1 & & & & & & \\ & b''_3 & & & pv''_{3(k+1)} & \cdots & pv''_{3l} \\ & & \ddots & & \vdots & \ddots & \vdots \\ & & & b''_k & pv''_{k(k+1)} & \cdots & pv''_{kl} \\ & & & & \vdots & \ddots & \vdots \\ & & & & pv''_{l(k+1)} & \cdots & pv''_{ll} \end{pmatrix} = \begin{pmatrix} 1 & \\ & A_7 \end{pmatrix},$$

where $A_7 \in K[x,y]^{(l-2)\times(l-2)}$. It follows that

$$A_2 \sim f_1 \cdot \begin{pmatrix} 1 & \\ & A_4 \end{pmatrix} \sim \begin{pmatrix} f_1 & \\ & f_2 \boldsymbol{I}_{l-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ & A_5 \end{pmatrix} \sim \begin{pmatrix} f_1 & \\ & f_2 \boldsymbol{I}_{l-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & A_7 \end{pmatrix}.$$

Repeating the above calculation process, we derive the following equivalence relation:

$$A_2 \sim \mathrm{diag}\{f_1, \cdots, f_k, \underbrace{f_k, \cdots, f_k}_{l-k}\} \cdot \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & pc_{(k+1)(k+1)} & \cdots & pc_{(k+1)l} \\ & & & \vdots & \ddots & \vdots \\ & & & pc_{l(k+1)} & \cdots & pc_{ll} \end{pmatrix},$$

where $c_{ij} \in K[x]$ for $k+1 \le i, j \le l$. Let $C = (c_{ij}) \in K[x]^{(l-k)\times(l-k)}$. Then

$$A_2 \sim \mathrm{diag}\{f_1, \cdots, f_k, f_k p, \cdots, f_k p\} \cdot \begin{pmatrix} \boldsymbol{I}_k & \\ & C \end{pmatrix}.$$

According to Lemmas 2.10 and 3.6, we have $d_j(A_2) = f_1 \cdots f_k (f_k p)^{j-k} d_{j-k}(C)$ for $k+1 \leq j \leq l$. Then

$$d_{j-k}(C) = \frac{f_{k+1} \cdots f_j}{f_k^{j-k}}, \qquad \text{where } j = k+1, \cdots, l.$$

It follows that the Smith form of C is

$$S_C = \begin{pmatrix} \frac{f_{k+1}}{f_k} & & & \\ & \frac{f_{k+2}}{f_k} & & \\ & & \ddots & \\ & & & \frac{f_l}{f_k} \end{pmatrix}.$$

This implies that there are two unimodular matrices $U_C, V_C \in K[x]^{(l-k)\times(l-k)}$ such that $U_C C V_C = S_C$. Since

$$\begin{aligned} &\begin{pmatrix} \mathrm{diag}\{f_1, \cdots, f_k\} & \\ & f_k p \boldsymbol{I}_{l-k} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{I}_k & \\ & C \end{pmatrix} \\ = &\begin{pmatrix} \boldsymbol{I}_k & \\ & U_C^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{diag}\{f_1, \cdots, f_k\} & \\ & f_k p S_C \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{I}_k & \\ & V_C^{-1} \end{pmatrix}, \end{aligned}$$

we have

$$\mathrm{diag}\{f_1, \cdots, f_k, f_k p, \cdots, f_k p\} \cdot \begin{pmatrix} \boldsymbol{I}_k & \\ & C \end{pmatrix} \sim \mathrm{diag}\{f_1, \cdots, f_k, f_{k+1} p, \cdots, f_l p\}.$$

It follows that

$$A_1 \sim A_2 \sim \mathrm{diag}\{f_1, \cdots, f_k, f_{k+1} p, \cdots, f_l p\}.$$

Second: $s_1 < s_2 < \cdots < s_k < s$.

According to Lemma 3.7, we have $d_i(A) = f_1 \cdots f_i p^{s_1 + \cdots + s_i}$ for $1 \leq i \leq k$, and $d_j(A) = f_1 \cdots f_j p^{s_1 + \cdots + s_k + (j-k)(s+1)}$ for $k+1 \leq j \leq l$. Let $s' = s - s_1$, and $s_i' = s_i - s_1$, where $i = 2, \cdots, k$. Assume that $U = (u_{ij})_{l \times l}$, then

$$A = f_1 p^{s_1} \cdot \begin{pmatrix} u_{11} & \cdots & u_{1k} & p u_{1(k+1)} & \cdots & p u_{1l} \\ \frac{f_2}{f_1} p^{s_2'} u_{21} & \cdots & \frac{f_2}{f_1} p^{s_2'} u_{2k} & \frac{f_2}{f_1} p^{s_2'+1} u_{2(k+1)} & \cdots & \frac{f_2}{f_1} p^{s_2'+1} u_{2l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{f_k}{f_1} p^{s_k'} u_{k1} & \cdots & \frac{f_k}{f_1} p^{s_k'} u_{kk} & \frac{f_k}{f_1} p^{s_k'+1} u_{k(k+1)} & \cdots & \frac{f_k}{f_1} p^{s_k'+1} u_{kl} \\ \frac{f_{k+1}}{f_1} p^{s'} u_{(k+1)1} & \cdots & \frac{f_{k+1}}{f_1} p^{s'} u_{(k+1)k} & \frac{f_{k+1}}{f_1} p^{s'+1} u_{(k+1)(k+1)} & \cdots & \frac{f_{k+1}}{f_1} p^{s'+1} u_{(k+1)l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{f_l}{f_1} p^{s'} u_{l1} & \cdots & \frac{f_l}{f_1} p^{s'} u_{lk} & \frac{f_l}{f_1} p^{s'+1} u_{l(k+1)} & \cdots & \frac{f_l}{f_1} p^{s'+1} u_{ll} \end{pmatrix}.$$

Letting $A = f_1 p^{s_1} \cdot A_1$, we have $J_1(A) = I_1(A_1)$. We assert that $\langle u_{11}, \cdots, u_{1k} \rangle = 1$. Otherwise, there exists a point $x_2 \in \mathfrak{K}$ such that $u_{1i}(x_2) = 0$, where $i = 1, \cdots, k$. This implies that $(x_2, g(x_2)) \in \mathbb{V}(I_1(A_1))$. This contradicts the fact that $I_1(A_1) = J_1(A) = K[x, y]$. Then there is a unimodular matrix $U_1 \in K[x]^{k\times k}$ such that $(u_{11}, \cdots, u_{1k}) \cdot U_1 = (1, 0, \cdots, 0)$. Let $U_2 = \begin{pmatrix} U_1 & \\ & \boldsymbol{I}_{l-k} \end{pmatrix}$ and $A_2 = A_1 \cdot U_2$. Then $U_2 \in K[x]^{l\times l}$ is unimodular and

$$A_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 & pu_{1(k+1)} & \cdots & pu_{1l} \\ \frac{f_2}{f_1}p^{s'_2}u'_{21} & \frac{f_2}{f_1}p^{s'_2}u'_{22} & \cdots & \frac{f_2}{f_1}p^{s'_2}u'_{2k} & \frac{f_2}{f_1}p^{s'_2+1}u_{2(k+1)} & \cdots & \frac{f_2}{f_1}p^{s'_2+1}u_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{f_k}{f_1}p^{s'_k}u'_{k1} & \frac{f_k}{f_1}p^{s'_k}u'_{k2} & \cdots & \frac{f_k}{f_1}p^{s'_k}u'_{kk} & \frac{f_k}{f_1}p^{s'_k+1}u_{k(k+1)} & \cdots & \frac{f_k}{f_1}p^{s'_k+1}u_{kl} \\ \frac{f_{k+1}}{f_1}p^{s'}u'_{(k+1)1} & \frac{f_{k+1}}{f_1}p^{s'}u'_{(k+1)2} & \cdots & \frac{f_{k+1}}{f_1}p^{s'}u'_{(k+1)k} & \frac{f_{k+1}}{f_1}p^{s'+1}u_{(k+1)(k+1)} & \cdots & \frac{f_{k+1}}{f_1}p^{s'+1}u_{(k+1)l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{f_l}{f_1}p^{s'}u'_{l1} & \frac{f_l}{f_1}p^{s'}u'_{l2} & \cdots & \frac{f_l}{f_1}p^{s'}u'_{lk} & \frac{f_l}{f_1}p^{s'+1}u_{l(k+1)} & \cdots & \frac{f_l}{f_1}p^{s'+1}u_{ll} \end{pmatrix}.$$

Let

$$U_3 = \begin{pmatrix} 1 & & & \\ -\frac{f_2}{f_1}p^{s'_2}u'_{21} & 1 & & \\ \vdots & & \ddots & \\ -\frac{f_l}{f_1}p^{s'}u'_{l1} & & & 1 \end{pmatrix} \text{ and } U_4 = \begin{pmatrix} 1 & & -pu_{1(k+1)} & \cdots & -pu_{1l} \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

where $U_3, U_4 \in K[x,y]^{l\times l}$ are unimodular matrices. Letting $A_3 = U_3A_2U_4$, we get

$$A_3 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \frac{f_2}{f_1}p^{s'_2}u'_{22} & \cdots & \frac{f_2}{f_1}p^{s'_2}u'_{2k} & \frac{f_2}{f_1}p^{s'_2+1}u'_{2(k+1)} & \cdots & \frac{f_2}{f_1}p^{s'_2+1}u'_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{f_k}{f_1}p^{s'_k}u'_{k2} & \cdots & \frac{f_k}{f_1}p^{s'_k}u'_{kk} & \frac{f_k}{f_1}p^{s'_k+1}u'_{k(k+1)} & \cdots & \frac{f_k}{f_1}p^{s'_k+1}u'_{kl} \\ 0 & \frac{f_{k+1}}{f_1}p^{s'}u'_{(k+1)2} & \cdots & \frac{f_{k+1}}{f_1}p^{s'}u'_{(k+1)k} & \frac{f_{k+1}}{f_1}p^{s'+1}u'_{(k+1)(k+1)} & \cdots & \frac{f_{k+1}}{f_1}p^{s'+1}u'_{(k+1)l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{f_l}{f_1}p^{s'}u'_{l2} & \cdots & \frac{f_l}{f_1}p^{s'}u'_{lk} & \frac{f_l}{f_1}p^{s'+1}u'_{l(k+1)} & \cdots & \frac{f_l}{f_1}p^{s'+1}u'_{ll} \end{pmatrix}.$$

It follows from the form of A_3 that $A_3 = \begin{pmatrix} 1 & \\ & \frac{f_2}{f_1}p^{s'_2}\boldsymbol{I}_{l-1} \end{pmatrix} \cdot A_4$, where

$$A_4 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & u'_{22} & \cdots & u'_{2k} & pu'_{2(k+1)} & \cdots & pu'_{2l} \\ 0 & \frac{f_3}{f_2}p^{s''_3}u'_{32} & \cdots & \frac{f_3}{f_2}p^{s''_3}u'_{3k} & \frac{f_3}{f_2}p^{s''_3+1}u'_{3(k+1)} & \cdots & \frac{f_3}{f_2}p^{s''_3+1}u'_{3l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{f_k}{f_2}p^{s''_k}u'_{k2} & \cdots & \frac{f_k}{f_2}p^{s''_k}u'_{kk} & \frac{f_k}{f_2}p^{s''_k+1}u'_{k(k+1)} & \cdots & \frac{f_k}{f_2}p^{s''_k+1}u'_{kl} \\ 0 & \frac{f_{k+1}}{f_2}p^{s''}u'_{(k+1)2} & \cdots & \frac{f_{k+1}}{f_2}p^{s''}u'_{(k+1)k} & \frac{f_{k+1}}{f_2}p^{s''+1}u'_{(k+1)(k+1)} & \cdots & \frac{f_{k+1}}{f_2}p^{s''+1}u'_{(k+1)l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{f_l}{f_2}p^{s''}u'_{l2} & \cdots & \frac{f_l}{f_2}p^{s''}u'_{lk} & \frac{f_l}{f_2}p^{s''+1}u'_{l(k+1)} & \cdots & \frac{f_l}{f_2}p^{s''+1}u'_{ll} \end{pmatrix},$$

$s''_i = s'_i - s'_2$ for $i = 3, \cdots, k$, and $s'' = s' - s'_2$. Through the above calculations and reductions, we obtain

$$A = U_3^{-1} \cdot f_1p^{s_1} \cdot \begin{pmatrix} 1 & \\ & \frac{f_2}{f_1}p^{s'_2}\boldsymbol{I}_{l-1} \end{pmatrix} \cdot A_4 \cdot U_4^{-1} \cdot U_2^{-1}.$$

It follows that

$$A \sim \begin{pmatrix} f_1 p^{s_1} & \\ & f_2 p^{s_2} \boldsymbol{I}_{l-1} \end{pmatrix} \cdot A_4.$$

Repeating the above calculation process, we derive the following equivalence relation:

$$A \sim \begin{pmatrix} f_1p^{s_1} & & & \\ & \ddots & & \\ & & f_kp^{s_k} & \\ & & & f_kp^{s_k}\boldsymbol{I}_{l-k} \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \frac{f_{k+1}}{f_k}p^{s-s_k+1}v_{(k+1)(k+1)} & \cdots & \frac{f_{k+1}}{f_k}p^{s-s_k+1}v_{(k+1)l} \\ & & & \vdots & \ddots & \vdots \\ & & & \frac{f_l}{f_k}p^{s-s_k+1}v_{l(k+1)} & \cdots & \frac{f_l}{f_k}p^{s-s_k+1}v_{ll} \end{pmatrix},$$

where $v_{ij} \in K[x]$ for $k+1 \leq i, j \leq l$. Letting $V = \begin{pmatrix} v_{(k+1)(k+1)} & \cdots & v_{(k+1)l} \\ \vdots & \ddots & \vdots \\ v_{l(k+1)} & \cdots & v_{ll} \end{pmatrix}$, we have

$$A \sim \mathrm{diag}\{f_1p^{s_1}, \cdots, f_kp^{s_k}, f_{k+1}p^{s+1}, \cdots, f_lp^{s+1}\} \cdot \begin{pmatrix} \boldsymbol{I}_k & \\ & V \end{pmatrix}.$$

By the fact that $d_l(A) = f_1 \cdots f_l p^{s_1+\cdots+s_k+(l-k)(s+1)}$, we get that $\begin{pmatrix} \boldsymbol{I}_k & \\ & V \end{pmatrix}$ is a unimodular matrix. Therefore,

$$A \sim \mathrm{diag}\{f_1p^{s_1}, \cdots, f_kp^{s_k}, f_{k+1}p^{s+1}, \cdots, f_lp^{s+1}\}.$$

Third: The remaining cases except the first and second. The same conclusion can be derived through the above method.

Therefore, the proof is completed. ∎

Lemma 3.9 solves the second challenge raised in Example 3.1.

3.3 The Solution for Problem 2.7

Building upon the resolution of the above three challenges, we now first consider the case where all the square matrices in $\mathcal{M}$ are of full rank, and then extend it to the general case, so as to achieve the goal of completely solving Problem 2.7.

Theorem 3.10 *Let $F \in K[x, y]^{l\times l}$ with $d_l(F) = fp^t$, where $f \in K[x]$ and $t \in \mathbb{N}$. Then F is equivalent to its Smith form if and only if $J_i(F) = K[x, y]$ for $i = 1, \cdots, l$.*

Proof Without loss of generality, assume that the Smith form of F is

$$S = \mathrm{diag}\{f_1p^{s_1}, f_2p^{s_2}, \cdots, f_lp^{s_l}\},$$

where $f_1, \cdots, f_l \in K[x]$ satisfy $f_1 \cdots f_l = f$ and $f_1 \mid f_2 \mid \cdots \mid f_l$, and $s_1, \cdots, s_l \in \mathbb{N}$ satisfy $0 \leq s_1 \leq \cdots \leq s_l$ and $s_1 + \cdots + s_l = t$.

Necessity It is easy to verify that $J_i(S) = K[x, y]$ for $i = 1, \cdots, l$. Since $F \sim S$, It follows from Lemma 2.10 that $J_i(F) = J_i(S)$ and $J_i(F) = K[x, y]$, where $i = 1, \cdots, l$.

Sufficiency According to the primitive factorization theorem, there exist two polynomial matrices $F_1, F_2 \in K[x, y]^{l\times l}$ such that $F = F_1F_2$ with $\det(F_1) = f$. It follows from $\det(F) =$

$\det(F_1) \cdot \det(F_2)$ that $\det(F_2) = p^t$ and $\gcd(\det(F_1), \det(F_2)) = 1$. Based on Lemma 2.12, we have $d_i(F) = d_i(F_1) \cdot d_i(F_2)$ and $J_i(F_1) = J_i(F_2) = K[x, y]$, where $i = 1, \cdots, l$. Since $d_i(F) = f_1 \cdots f_i p^{s_1+\cdots+s_i}$ for $i = 1, \cdots, l$, we obtain

$$d_i(F_1) = f_1 \cdots f_i \text{ and } d_i(F_2) = p^{s_1+\cdots+s_i},$$

where $i = 1, \cdots, l$. This implies that the Smith form of F_1 is

$$S_{F_1} = \mathrm{diag}\{f_1, f_2, \cdots, f_l\}.$$

It follows from Lemma 2.16 that $F_1 \sim S_{F_1}$. Then, there are two unimodular matrices $U_1, U_2 \in K[x, y]$ such that

$$F_1 = U_1 S_{F_1} U_2. \tag{9}$$

Let $F_3 = U_2 F_2$, we get $d_i(F_3) = d_i(F_2)$ by the fact that U_2 is unimodular, where $i = 1, \cdots, l$. Since $d_1(F_3) = p^{s_1}$, there exists a polynomial matrix $F_4 \in K[x, y]^{l\times l}$ such that $F_3 = p^{s_1} F_4$. It follows from Equation (9) that

$$F \sim \mathrm{diag}\{f_1 p^{s_1}, f_2 p^{s_1}, \cdots, f_l p^{s_1}\} \cdot F_4. \tag{10}$$

If $s_2 = s_1$, then $F \sim \mathrm{diag}\{f_1 p^{s_1}, f_2 p^{s_2}, \cdots, f_l p^{s_2}\} \cdot F_4$. If $s_2 > s_1$, then by Lemma 3.4 we have $\mathrm{rank}(\phi_p(F_4)) = 1$. According to Lemma 3.2, there exist $F_5 \in K[x, y]^{l\times l}$ and a unimodular matrix $U_3 \in K[x]^{l\times l}$ such that

$$F_4 = U_3 \cdot \mathrm{diag}\{1, \underbrace{p, \cdots, p}_{l-1}\} \cdot F_5. \tag{11}$$

Combining Equations (10) and (11), we have

$$F \sim \mathrm{diag}\{f_1 p^{s_1}, f_2 p^{s_1}, \cdots, f_l p^{s_1}\} \cdot U_3 \cdot \mathrm{diag}\{1, p, \cdots, p\} \cdot F_5. \tag{12}$$

Let $F_6 = \mathrm{diag}\{f_1 p^{s_1}, f_2 p^{s_1}, \cdots, f_l p^{s_1}\} \cdot U_3 \cdot \mathrm{diag}\{1, p, \cdots, p\}$. By Lemma 3.7 we have $d_1(F_6) = f_1 p^{s_1}$. It follows from Lemma 2.11 that $J_1(F_6) = K[x, y]$. Based on Lemma 3.9, we obtain

$$F_6 \sim \mathrm{diag}\{f_1 p^{s_1}, f_2 p^{s_1+1}, \cdots, f_l p^{s_1+1}\}.$$

Then there are two unimodular matrices $U_4, U_5 \in K[x, y]^{l\times l}$ such that

$$F_6 = U_4 \cdot \mathrm{diag}\{f_1 p^{s_1}, f_2 p^{s_1+1}, \cdots, f_l p^{s_1+1}\} \cdot U_5.$$

Let $F_7 = U_5 F_5$, and it follows from Equation (12) that

$$F \sim \mathrm{diag}\{f_1 p^{s_1}, f_2 p^{s_1+1}, \cdots, f_l p^{s_1+1}\} \cdot F_7.$$

Repeating the above calculation process, we will obtain the following equivalence relation within a finite number of steps:

$$F \sim \mathrm{diag}\{f_1 p^{s_1}, f_2 p^{s_2}, \cdots, f_l p^{s_l}\} \cdot F_N,$$

where $F_N \in K[x,y]^{l\times l}$. It is easy to check that F_N is a unimodular matrix. Therefore, we have

$$F \sim \mathrm{diag}\{f_1 p^{s_1}, f_2 p^{s_2}, \cdots, f_l p^{s_l}\}.$$

The proof is completed. ∎

Corollary 3.11 *Let $F \in K[x,y]^{l\times m}$ with $d_r(F) = fp^t$, where $r = \mathrm{rank}(F)$, $f \in K[x]$ and $t \in \mathbb{N}$. Then F is equivalent to its Smith form if and only if $J_i(F) = K[x,y]$ for $i = 1, \cdots, r$.*

Proof Without loss of generality, assume that the Smith form of F is

$$S = \begin{pmatrix} \mathrm{diag}\{f_1 p^{s_1}, \cdots, f_r p^{s_r}\} & 0_{r\times(m-r)} \\ 0_{(l-r)\times r} & 0_{(l-r)\times(m-r)} \end{pmatrix},$$

where $f_1, \cdots, f_r \in K[x]$ satisfy $f_1 \cdots f_r = f$ and $f_1 \mid f_2 \mid \cdots \mid f_r$, and $s_1, \cdots, s_r \in \mathbb{N}$ satisfy $0 \le s_1 \le \cdots \le s_r$ and $s_1 + \cdots + s_r = t$.

Necessity It is easy to verify that $J_i(S) = K[x,y]$ for $i = 1, \cdots, r$. Since $F \sim S$, it follows from Lemma 2.10 that $J_i(F) = J_i(S)$ and $J_i(F) = K[x,y]$, where $i = 1, \cdots, r$.

Sufficiency Since $J_r(F) = K[x,y]$, it follows from Lemma 2.14 that there exist a polynomial matrix $G_1 \in K[x,y]^{l\times r}$ and $F_1 \in K[x,y]^{r\times m}$ such that $F = G_1F_1$ with F_1 being a ZLP matrix. According to Corollary 2.13, there is a unimodular matrix $U \in K[x,y]^{m\times m}$ such that $F_1U = (\boldsymbol{I}_r, 0_{r\times(m-r)})$. It follows that $F \sim (G_1, 0_{l\times(m-r)})$. By Lemma 2.10, we have $J_r(G_1) = J_r(F) = K[x,y]$. Using Lemma 2.14 again, there exist $G_2 \in K[x,y]^{l\times r}$ and $G_3 \in K[x,y]^{r\times r}$ such that $G_1 = G_2G_3$ with G_2 being a ZRP matrix. Based on Corollary 2.13, there is a unimodular matrix $V \in K[x,y]^{l\times l}$ such that $VG_2 = \begin{pmatrix} \boldsymbol{I}_r \\ 0_{(l-r)\times r} \end{pmatrix}$. Then we obtain

$$F \sim \begin{pmatrix} G_3 & 0_{r\times(m-r)} \\ 0_{(l-r)\times r} & 0_{(l-r)\times(m-r)} \end{pmatrix}.$$

It follows from Lemma 2.10 that

$$d_i(G_3) = d_i(F) = f_1 \cdots f_i p^{s_1+\cdots+s_i} \text{ and } J_i(G_3) = J_i(F) = K[x,y],$$

where $i = 1, \cdots, r$. According to Theorem 3.10, there are two unimodular matrices $V_1, U_1 \in K[x,y]^{r\times r}$ such that

$$G_3 = V_1 \cdot \mathrm{diag}\{f_1 p^{s_1}, \cdots, f_r p^{s_r}\} \cdot U_1.$$

Let $V_2 = \begin{pmatrix} V_1 & \\ & \boldsymbol{I}_{l-r} \end{pmatrix}$ and $U_2 = \begin{pmatrix} U_1 & \\ & \boldsymbol{I}_{m-r} \end{pmatrix}$, where $V_2 \in K[x,y]^{l\times l}$ and $U_2 \in K[x,y]^{m\times m}$. Obviously, V_2, U_2 are two unimodular matrices. This implies that

$$F \sim \begin{pmatrix} \mathrm{diag}\{f_1 p^{s_1}, \cdots, f_r p^{s_r}\} & 0_{r\times(m-r)} \\ 0_{(l-r)\times r} & 0_{(l-r)\times(m-r)} \end{pmatrix}.$$

The proof is completed. ∎

4 Conclusions

This paper focuses on the equivalence problem between a class of bivariate polynomial matrices and their Smith forms. Let $F \in K[x,y]^{l\times l}$ be of full rank, and $d_l(F) = f(x)(y-g(x))^t$, where $f, g \in K[x]$. The main idea is as follows. First, we use the primitive factorization theorem to factorize F into the product of two matrices whose determinants are coprime. Then, by taking advantage of the special form of $y - g(x)$, we construct a homomorphism from $K[x,y]$ to $K[x]$. Consequently, we can use the properties of the Euclidean domain $K[x]$ to reduce F to its Smith form. For the cases of non-square matrices or matrices that are not of full rank, we resort to the Quillen-Suslin theorem to transform them into the cases of square matrices with full rank, thereby completely resolving Problem 2.7.

Based on the research presented in this paper, we naturally pose the following problem. Let $F \in K[x,y]$ with rank r, what is the necessary and sufficient condition for F to be equivalent to its Smith form? At this point, $d_r(F)$ no longer has a special form. Can new techniques be developed to address this problem? This problem warrants further investigation.

Conflict of Interest

The authors declare no conflict of interest.

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