Hypergeometric Series Solutions of Linear Operator Equations

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Background
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Abramov, Paule and Petkovšek visited formal power series solutions and basic hypergeometric series solutions for \( q \)-difference equations.

\[ y(x) = \sum_{k=0}^{\infty} c(q^k)x^k. \]
Our aim

Given a linear differential/difference equation

\[ L(y(x)) = 0, \]

find a hypergeometric series solution

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\[ (1 - x^2)p''(x) - xp'(x) + n^2 p(x) = 0 \]

\[ \Rightarrow p(x) = \sum_{k=0}^{n} \frac{(-n)_k (n)_k}{(1/2)_k k!} \left( \frac{1 - x}{2} \right)^k = 2F_1 \left( \begin{array}{c} -n, n \\ 1/2 \end{array} \middle| \frac{1 - x}{2} \right). \]
Solving linear operator equations
Suitable bases

Let $L$ be a linear operator acting on the ring $K[x]$. 
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where $A_k, B_k \in K$ and $h$ is a fixed positive integer.
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We further require that

- $b_k(x)$ are monic
- $b_{k-1}(x)$ divides $b_k(x)$
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Then $b_k(x) = (x - x_1)(x - x_2) \cdots (x - x_k)$. 
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Such bases are called suitable bases.
Searching for suitable bases

We solve $x_1, \ldots, x_k$ for explicit integer $k$. Recall that

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$$A_k = [x^k]L(b_k(x)) \quad \text{and} \quad B_k = [x^{k-h}](L(b_k(x)) - A_k b_k(x))$$

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Comparing coefficients of \( x^i \), we obtain a system of polynomial equations on \( x_1, \ldots, x_k \).

Starting from \( k = 1 \), we iteratively set up and solve the equations until reaching a certain degree \( k_0 \).
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Starting from $k = 1$, we iteratively set up and solve the equations until reaching a certain degree $k_0$.

Finally, guess the general form of $x_k$ from the pattern.
Example

\[ L(p(x)) = (1 - x^2)p''(x) - xp'(x) + n^2 p(x). \]
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Take \( h = 1 \) and set

\[ b_0(x) = 1, \quad b_1(x) = x - x_1. \]
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We do not obtain any equation on \( x_1 \).
Set $b_2(x) = (x - x_1)(x - x_2)$. 
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\[
(n^2 - 4)x^2 - (n^2 - 1)(x_1 + x_2)x + 2 + n^2x_1x_2
= A_2(x - x_1)(x - x_2) + B_2(x - x_1),
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\[A_2 = n^2 - 4, \quad B_2 = -3(x_1 + x_2), \quad \text{and} \quad x_1x_2 = 3x_1^2 - 2.\]
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For $k \geq 3$, we derive

$$x_1 = x_2 = \cdots = x_k = 1 \quad \text{and} \quad x_1 = x_2 = \cdots = x_k = -1.$$
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x_1 = x_2 = \cdots = x_k = 1 \quad \text{and} \quad x_1 = x_2 = \cdots = x_k = -1.
\]

Guess:

\[
b_k(x) = (x + 1)^k \quad \text{or} \quad b_k(x) = (x - 1)^k.
\]
Verify suitable bases

\[ L(b_k(x)) = A_k b_k(x) + B_k b_{k-h}(x) \]
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holds if and only if

\[ A_k = [x^h] \frac{L(b_k(x))}{b_{k-h}(x)}, \quad \text{and} \quad B_k = [x^0] \left( \frac{L(b_k(x))}{b_{k-h}(x)} - A_k \frac{b_k(x)}{b_{k-h}(x)} \right), \]

and

\[ \frac{L(b_k(x))}{b_{k-h}(x)} = A_k \frac{b_k(x)}{b_{k-h}(x)} + B_k. \]
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- Verify
- solve out \( A_k \) and \( B_k \)
Example

Let

\[ L(p(x)) = (1 - x^2)p''(x) - xp'(x) + n^2 p(x). \]

Verify \((x - 1)^k\).

It is a suitable basis and

\[ A_k = n^2 - k^2 \quad \text{and} \quad B_k = k - 2k^2. \]
As done by Abramov and Petkovšek, $L$ can be extended to formal series of the form $\sum_{k=0}^{\infty} c_k b_k(x)$ by setting

$$L \left( \sum_{k=0}^{\infty} c_k b_k(x) \right) = \sum_{k=0}^{\infty} (c_k A_k + c_{k+h} B_{k+h}) b_k(x).$$
Series solutions

As done by Abramov and Petkovšek, $L$ can be extended to formal series of the form $\sum_{k=0}^{\infty} c_k b_k(x)$ by setting

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Suppose $c_k A_k + c_{k+h} B_{k+h} = 0$, $\forall k \in \mathbb{N}$.

Then $y(x) = \sum_{k=0}^{\infty} c_k b_k(x)$ is a formal solution to the equation $L(y(x)) = 0$. 
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Then \( y(x) = \sum_{k=0}^{\infty} c_k b_k(x) \) is a formal solution to the equation \( L(y(x)) = 0 \).

When \( \sum_{k=0}^{\infty} c_k b_k(x) \) is a finite summation, it is a real solution.
Hypergeometric series solutions

When $A_k, B_k$ and $x_k$ are all rational functions of $k$, $t_k = c_k b_k(x)$ is an $h$-fold hypergeometric term.
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\frac{t_{k+h}}{t_k} = -\frac{A_k \cdot b_{k+h}(x)}{B_{k+h} \cdot b_k(x)}.
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\]

\[
\frac{t_{k+1}}{t_k} = -\frac{A_k}{B_{k+1}} (x - 1) = \frac{(k - n)(k + n)}{(k + 1)(k + 1/2)} \cdot \frac{1 - x}{2},
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Hypergeometric series solutions

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\[
y(x) = \sum_{k=0}^{\infty} t_k = t_0 \cdot _2F_1\left(\begin{array}{c}-n, n \\ 1/2\end{array}\right| \frac{1 - x}{2}\right)
\]
Differential/Difference equations
$L(p(x)) = (1-x^2)p''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)p'(x) + n(n + \alpha + \beta + 1)p(x)$.
Jacobi polynomials

\[ L(p(x)) = (1-x^2)p''(x) + (\beta-\alpha-(\alpha+\beta+2)x)p'(x) + n(n+\alpha+\beta+1)p(x). \]

\[ b_k: (x - 1)^k \text{ or } (x + 1)^k. \]
Jacobi polynomials

\[ L(p(x)) = (1-x^2)p''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)p'(x) + n(n + \alpha + \beta + 1)p(x). \]

\[ b_k: (x - 1)^k \text{ or } (x + 1)^k. \]

\[ P_{\alpha, \beta}^n(x) = \frac{(\alpha + 1)n}{n!} {}_2F_1 \left( \begin{array}{c} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{array} \right| \frac{1-x}{2} \right) \]

\[ = (-1)^n \frac{(\beta + 1)n}{n!} {}_2F_1 \left( \begin{array}{c} -n, n + \alpha + \beta + 1 \\ \beta + 1 \end{array} \right| \frac{1+x}{2} \right). \]
Hahn polynomials

\[ L(p(x)) = B(x)y(x+1) - (n(n+\alpha+\beta+1)+B(x)+D(x))y(x) + D(x)y(x-1), \]

where \( B(x) = (x + \alpha + 1)(x - N) \) and \( D(x) = x(x - \beta - N - 1) \).
Hahn polynomials

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where \( B(x) = (x + \alpha + 1)(x - N) \) and \( D(x) = x(x - \beta - N - 1) \).

\( b_k: \)

\{ (x+\alpha+1)_k \}, \quad \{ (-1)^{k}(-x+N+\beta+1)_k \}, \quad \{ (x-N)_k \}, \quad \{ (-1)^{k}(-x)_k \}.  

Hahn polynomials

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where \( B(x) = (x + \alpha + 1)(x - N) \) and \( D(x) = x(x - \beta - N - 1) \).

\( b_k:\)

\[ \{(x+\alpha+1)_k\}, \quad \{(-1)^k(-x+N+\beta+1)_k\}, \quad \{(x-N)_k\}, \quad \{(-1)^k(-x)_k\}. \]

\[ Q_n(x) = c_n \cdot {}_3F_2 \left( \begin{array}{c} -n, n + \alpha + \beta + 1, x + \alpha + 1 \\ \alpha + 1, \alpha + \beta + N + 2 \end{array} \right| 1 \right). \]
$x = x(s)$ and $L$ acts on $s$.

$L(p(s)) = B(s)p(s+1) - (n(n+\alpha+\beta+1)+B(s)+D(s))p(s)+D(s)p(s-1)$,

$B(s)$ and $D(s)$ are rational functions.
Non-uniform lattice

\[ x = x(s) \text{ and } L \text{ acts on } s. \]

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\[ B(s) \text{ and } D(s) \text{ are rational functions.} \]

\[ b_k(s) = (x(s) - x_1)(x(s) - x_2) \cdots (x(s) - x_k) \]
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\[ b_k(s) = (x(s) - x_1)(x(s) - x_2) \cdots (x(s) - x_k) \]

For Racah polynomials, we find \( x(s) = s(s + \gamma + \delta + 1) \) and \( x_k = x(s_k) \):

\[ s_k = k+\alpha-\gamma-\delta-1, \quad s_k = k-\delta-1, \quad s_k = k-1, \quad \text{or} \quad s_k = k+\beta-\gamma-1. \]
Non-uniform lattice

\[ x = x(s) \text{ and } L \text{ acts on } s. \]

\[ L(p(s)) = B(s)p(s+1) - (n(n+\alpha+\beta+1) + B(s) + D(s))p(s) + D(s)p(s-1), \]

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\[ s_k = k + \alpha - \gamma - \delta - 1, \quad s_k = k - \delta - 1, \quad s_k = k - 1, \quad \text{or} \quad s_k = k + \beta - \gamma - 1. \]

\[ R_n(x(s)) = \binom{4F3}{-n, n + \alpha + \beta + 1, -s + \alpha - \gamma - \delta, s + \alpha + 1}{\alpha + 1, \alpha - \delta + 1, \alpha + \beta - \gamma + 1}. \]
Recurrence relations
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\[ xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x). \]
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Let

\[ p(n) = \sum_{k=0}^{\infty} c_k b_k(n), \quad r(n) = a_{n+1}/a_n. \]
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Define \( L \) by

\[ L(p(n)) = \alpha_n r(n) p(n+1) + (\beta_n - x) p(n) + \frac{\gamma_n}{r(n-1)} p(n-1). \]
Choose $r(n)$

(a) $r(n) = p(n)/q(n)$, $q(n)$ is a factor of the numerator of $\alpha_n$ and $p(n-1)$ is a factor of the numerator of $\gamma_n$.

(b) The numerator of $\frac{\gamma_n}{r(n-1)}$ is divisible by $n$.

(c) $L(1)$ is a constant independent of $n$. 
\[ P_{n+1}(x) + (n - x)P_n(x) + \alpha n^2 P_{n-1}(x) = 0. \]
\[ P_{n+1}(x) + (n - x)P_n(x) + \alpha n^2 P_{n-1}(x) = 0. \]

\[ r(n) = \frac{-1 \pm \sqrt{1 - 4\alpha}}{2}(n + 1). \]
Example

\[ P_{n+1}(x) + (n - x)P_n(x) + \alpha n^2 P_{n-1}(x) = 0. \]

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\[ L(p(n)) = \frac{u - 1}{2}(n + 1)p(n + 1) + (n - x)p(n) - \frac{u + 1}{2}np(n - 1), \]

where \( u^2 = 1 - 4\alpha. \)
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where \( u^2 = 1 - 4\alpha \).

\[ b_k(n) = (-1)^k(-n)_k \]

and

\[ P_n(x) = a_0 \left( \frac{u - 1}{2} \right)^n n! \,_2F_1 \begin{pmatrix} -n, (-2x + u - 1)/2u \\ 1 \end{pmatrix} \begin{pmatrix} 2u \\ u - 1 \end{pmatrix}. \]
The Al-Salam-Chihara polynomials

$$2x Q_n(x) = Q_{n+1}(x) + (a + b)q^n Q_n(x) + (1 - q^n)(1 - abq^{n-1}) Q_{n-1}(x).$$
The Al-Salam-Chihara polynomials

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\[ b_k(t) = (t - 1)(t - q^{-1}) \cdots (t - q^{-k+1}) \quad \text{and} \]

\[ Q_n(x) = a_0 \frac{(ab; q)_n}{a^n} 3\phi_2 \left( \begin{array}{c} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{array} \left| q; q \right. \right), \quad x = \cos \theta. \]
Thanks for attending