Introduction to $D$-module Theory. Algorithms for Computing Bernstein-Sato Polynomials

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- Viktor Levandovskyy
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- Daniel Andres
Motivation to Singularity Theory
Motivation

- Let \( f \in \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial.
- \( p \in \mathbb{C}^n \) is said to be singular if \( p \in V(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \).

\[
\begin{align*}
\frac{\partial f}{\partial x}(p)(x - a) + \frac{\partial f}{\partial y}(p)(y - b) &= 0 \\
p &= (a, b) \\
f(p) &= 0 \\
X &= f^{-1}(0)
\end{align*}
\]

- To study singular points \( \leadsto \) invariants.
- Two hypersurfaces \( X = V(f), Y = V(g) \subseteq \mathbb{C}^n \) are called algebraically equivalent if there exists an algebraic isomorphism \( \varphi : X \to Y \).
First part of the talk

- Basic notations and definitions
- History of the problem
- ...
- Well-known properties.
- Algorithms for computing $b_f(s)$
Basic notations

- $\mathbb{C}$ the field of the complex numbers.

- $\mathbb{C}[s]$ the ring of polynomials in one variable over $\mathbb{C}$.

- $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ the ring of polynomials in $n$ variables.

- $D_n = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ the ring of $\mathbb{C}$-linear differential operators, i.e. the $n$-th Weyl algebra:
  \[ \partial_i x_i = x_i \partial_i + 1 \]

- $D_n[s] = D_n \otimes_\mathbb{C} \mathbb{C}[s]$. 
The Weyl algebra

\[ W_n = \mathbb{C} \left\langle \{ \phi x_1, \ldots, \phi x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_1} \} \right\rangle \subset \text{End}_\mathbb{C}(\mathbb{C}[x]) \]

\[ \mathbb{C}[x_1, \ldots, x_n] \rightarrow \mathbb{C}[x_1, \ldots, x_n] \]

\[ \phi x_i : f \mapsto x_i f \]

\[ \frac{\partial}{\partial x_i} : f \mapsto \frac{\partial f}{\partial x_i} \]

\[ D_n = \frac{\mathbb{C}\{x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\}}{\langle \{x_i x_j - x_j x_i, \partial_i \partial_j - \partial_j \partial_i, \partial_i x_j - x_j \partial_i - \delta_{ij}\} \rangle} \]
The Weyl algebra

\[ W_n = \mathbb{C} \left\langle \{ \phi x_1, \ldots, \phi x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_1} \} \right\rangle \subset \text{End}_\mathbb{C}(\mathbb{C}[x]) \]

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\[ D_n = \mathbb{C}\{x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\} \]

\[ \langle \{ x_i, x_j - x_j x_i, \partial_i \partial_j - \partial_j \partial_i, \partial_i x_j - x_j \partial_i - \delta_{ij} \} \rangle \]

Proposition

The natural map \( x_i \mapsto \phi x_i, \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial x_i} \) is a \( \mathbb{C} \)-algebra isomorphism between \( W_n \) and \( D_n \).
The non-commutative relations come from Leibniz rule.

\[ \partial_i x_i = x_i \partial_i + 1 \]

The set of monomials \( \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\} \) forms a basis as \( \mathbb{C} \)-vector space.

\[ P = \sum_{\alpha, \beta} a_{\alpha \beta} x^\alpha \partial^\beta \quad (a_{\alpha \beta} \in \mathbb{C}) \]

To define a \( D_n \)-module, it is enough to give the action over the generators and then check that the relations are preserved.

For any monomial order there exists a Gröbner basis.

The Weyl algebra is simple, i.e. there are no two-sided ideals.
Let $f \in \mathbb{C}[x]$ be a non-zero polynomial.

By $\mathbb{C}[x, s, \frac{1}{f}]$ we denote the ring of rational functions of the form

$$\frac{g(x, s)}{f^k}$$

where $g(x, s) \in \mathbb{C}[x, s] = \mathbb{C}[x_1, \ldots, x_n, s]$.

We denote by $\mathbb{C}[x, s, \frac{1}{f}] \cdot T$ the free $\mathbb{C}[x, s, \frac{1}{f}]$-module of rank one generated by the symbol $T$. 

$$G(x, s) \cdot T$$
The $D_n[s]$-module $\mathbb{C}[x, s, \frac{1}{f}] \cdot T$ (Bernstein, 1972)

- $\mathbb{C}[x, s, \frac{1}{f}] \cdot T$ has a natural structure of left $D_n[s]$-module.

\[
\begin{align*}
    x_i \cdot (G(x, s) \cdot T) & = x_i G(x, s) \cdot T \\
    \partial_i \cdot (G(x, s) \cdot T) & = \left( \frac{\partial G}{\partial x_i} + G(x, s) s \frac{\partial f}{\partial x_i} \frac{1}{f} \right) \cdot T \\
    s \cdot (G(x, s) \cdot T) & = sG(x, s) \cdot T
\end{align*}
\]
The previous expression defines an action.

\begin{itemize}
\item \((x_i x_j - x_j x_i) \bullet (G(x, s) \cdot T) = 0 \cdot T \)
\item \((\partial_i \partial_j - \partial_j \partial_i) \bullet (G(x, s) \cdot T) = 0 \cdot T \)
\item \((\partial_i x_j - x_j \partial_i) \bullet (G(x, s) \cdot T) = 0 \cdot T \quad (i \neq j) \)
\item \((\partial_i x_i - x_i \partial_i - 1) \bullet (G(x, s) \cdot T) = 0 \cdot T \)
\end{itemize}
Where does this action come from? \[ T = f^s \]
Where does this action come from?

\[ \partial_i \bullet (G(x, s) \cdot T) = \left( \frac{\partial G}{\partial x_i} + G(x, s) s \frac{\partial f}{\partial x_i} \frac{1}{f} \right) \cdot T \]

\[ \Downarrow \]

\[ \partial_i \bullet (G(x, s) \cdot f^s) = \left( \frac{\partial G}{\partial x_i} + G(x, s) s \frac{\partial f}{\partial x_i} \frac{1}{f} \right) \cdot f^s \]
Classical notation

- $\mathbb{C}[x, s, \frac{1}{f}] \cdot f^s := \mathbb{C}[x, s, \frac{1}{f}] \cdot T$

- $f^s := 1 \cdot T$

- $f^{s+k} := f^k \cdot T \quad (k \in \mathbb{Z})$

- $0 := 0 \cdot T$

\[
\partial_i \cdot f^s = s \frac{\partial f}{\partial x_i} f^{s-1} \quad \Rightarrow \quad \partial_i \cdot T = s \frac{\partial f}{\partial x_i} \frac{1}{f} \cdot T
\]
The global $b$-function

**Theorem (Bernstein, 1972)**

For every polynomial $f \in \mathbb{C}[x]$ there exists a non-zero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P(s) \in D_n[s]$ such that

$$P(s) \cdot f^{s+1} = b(s) \cdot f^s \in \mathbb{C}[x, s, \frac{1}{f}] \cdot f^s.$$ 

**Definition (Bernstein & Sato, 1972)**

The set of all possible polynomials $b(s)$ satisfying the previous equation is an ideal of $\mathbb{C}[s]$. The monic generator of this ideal is denoted by $b_f(s)$ and called the global Bernstein-Sato polynomial or global $b$-function.
\section*{Examples}

1. Normal crossing divisor $f = x_1^{m_1} \cdots x_k^{m_k}$.

\begin{align*}
P(s) &= c \cdot \partial_1^{m_1} \cdots \partial_k^{m_k} \\
b_f(s) &= \prod_{i_1=1}^{m_1} \left( s + \frac{i_1}{m_1} \right) \cdots \prod_{i_k=1}^{m_k} \left( s + \frac{i_k}{m_k} \right)
\end{align*}

2. The classical cusp $f = x^2 + y^3$.

\begin{align*}
P(s) &= \frac{1}{12} y \partial_x^2 \partial_y + \frac{1}{27} \partial^3 y + \frac{1}{4} \partial_x s + \frac{3}{8} \partial_x^2 \\
b_f(s) &= \left( s + 1 \right) \left( s + \frac{5}{6} \right) \left( s + \frac{7}{6} \right)
\end{align*}
Now assume that
- \( f \in \mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\} \) is a convergent power series.
- \( \mathcal{D}_n \) is the ring of differential operators with coefficients in \( \mathcal{O} \).

**Theorem (Kashiwara & Björk, 1976)**

For every \( f \in \mathcal{O} \) there exists a non-zero polynomial \( b(s) \in \mathbb{C}[s] \) and a differential operator \( P(s) \in \mathcal{D}_n[s] \) such that

\[
P(s) \cdot f^{s+1} = b(s) \cdot f^s \quad \in \quad \mathcal{O}[s, \frac{1}{f}] \cdot f^s.
\]

**Definition**

The monic polynomial in \( \mathbb{C}[s] \) of lowest degree which satisfies the previous equation is denoted by \( b_{f,0}(s) \) and called the **local Bernstein-Sato polynomial** of \( f \) at the origin or **local b-function**.
Some well-known properties of the $b$-function:

1. The $b$-function is always a multiple of $(s + 1)$. The equality holds if and only if $f$ is smooth.

2. The (resp. local) Bernstein-Sato polynomial is an (resp. analytic) algebraic invariant of the singularity $V = \{ f = 0 \}$.

3. The set $\{ e^{2\pi i \alpha} \mid b_{f,0}(\alpha) = 0 \}$ coincides with the eigenvalues of the monodromy of the Milnor fibration. (Malgrange, 1975 and 1983).

J. Martín-Morales (jorge@unizar.es)  Algorithms for Computing Bernstein-Sato Polynomials
Some well-known properties of the $b$-function

1. Every root of $b_f(s)$ is negative and rational. (Kashiwara, 1976).

2. The roots of $b_f(s)$ belong to the real interval $(-n, 0)$. (Varchenko, 1980; Saito, 1994).

3. $b_f(s) = \text{lcm}_{p \in \mathbb{C}^n} \{b_{f,p}(s)\} = \text{lcm}_{p \in \text{Sing}(f)} \{b_{f,p}(s)\}$ (Briançon-Maisonobe and Mebkhout-Narváez, 1991).
Algorithms for computing the $b$-function

1. Functional equation, $P(s)f \ast f^s = b_f(s) \cdot f^s$.

2. By definition, $(\text{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] = \langle b_f(s) \rangle$.

3. Now find a system of generator of the annihilator and proceed with the elimination.

<table>
<thead>
<tr>
<th>Annihilator</th>
<th>Elimination</th>
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<tbody>
<tr>
<td>Levandovskyy (2008)</td>
<td></td>
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Second part of the talk

- Partial solution: the checkRoot algorithm

- Applications:
  2. Integral roots of $b$-functions.
  3. Stratification associated with local $b$-functions without employing primary ideal decomposition.
Another idea for computing the $b$-function

1. Obtain an upper bound for $b_f(s)$: find $B(s) \in \mathbb{C}[s]$ such that $b_f(s)$ divides $B(s)$.

$$B(s) = \prod_{i=1}^{d} (s - \alpha_i)^{m_i}.$$ 

2. Check whether $\alpha_i$ is a root of the $b$-function.

3. Compute its multiplicity $m_i$.

Remark

There are some well-known methods to obtain such $B(s)$: Resolution of Singularities.
The main trick

- By definition, $(\text{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] = \langle b_f(s) \rangle$.
- $(\text{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] + \langle q(s) \rangle = \langle b_f(s), q(s) \rangle$, $q(s) \in \mathbb{C}[s]$
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**Proposition**

\[
(\text{Ann}_{D_n[s]}(f^s) + \langle f, q(s) \rangle) \cap \mathbb{C}[s] = \langle b_f(s), q(s) \rangle \\
= \langle \gcd(b_f(s), q(s)) \rangle
\]
The main trick

- By definition, \((\text{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] = \langle b_f(s) \rangle.\)
- \((\text{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] + \langle q(s) \rangle = \langle b_f(s), q(s) \rangle, \text{ } q(s) \in \mathbb{C}[s].\)

**Proposition**

\[
\text{(Ann}_{D_n[s]}(f^s) + \langle f, q(s) \rangle) \cap \mathbb{C}[s] = \langle b_f(s), q(s) \rangle
= \langle \gcd(b_f(s), q(s)) \rangle
\]

**Corollary**

- \(m_\alpha\) the multiplicity of \(\alpha\) as a root of \(b_f(-s)\).
- \(J_i = \text{Ann}_{D_n[s]}(f^s) + \langle f, (s + \alpha)^{i+1} \rangle \subseteq D_n[s].\)

The following conditions are equivalent:

1. \(m_\alpha > i.\)
2. \((s + \alpha)^i \notin J_i.\)
Algorithm (compute the multiplicity of $\alpha$ as a root of $b_f(-s)$)

Input: $I = \text{Ann}_{D_n[s]}(f^s)$, $f$ a polynomial in $R_n$, $\alpha$ in $\mathbb{Q}$;
Output: $m_\alpha$, the multiplicity of $\alpha$ as a root of $b_f(-s)$;
Algorithm (compute the multiplicity of $\alpha$ as a root of $b_f(-s)$)

Input: $I = \text{Ann}_{D_n[s]}(f^s)$, $f$ a polynomial in $R_n$, $\alpha$ in $\mathbb{Q}$; 
Output: $m_\alpha$, the multiplicity of $\alpha$ as a root of $b_f(-s)$;

for $i = 0$ to $n$ do
  1. $J := I + \langle f, (s + \alpha)^{i+1} \rangle$; \hfill $J_i \subseteq D_n[s]$
  2. $G$ a reduced Gröbner basis of $J$ w.r.t. any term ordering;
  3. $r$ normal form of $(s + \alpha)^i$ with respect to $G$;
  4. if $r = 0$ then \hfill $r = 0 \iff (s + \alpha)^i \in J_i$
      $m_\alpha = i$; \hfill leave the for block
      break
  end if
end for

return $m_\alpha$
The checkRoot algorithm

This algorithm is much faster, than the computation of the whole Bernstein polynomial via Gröbner bases because:

- No elimination ordering is needed.
- The element $(s + \alpha)^{i+1}$ seems to simplify tremendously the computation.
Applications

1. Computation of the $b$-functions via upper bounds.
   - Embedded resolutions.
   - Topologically equivalent singularities.
   - A’Campo’s formula.
   - Spectral numbers.

2. Integral roots of $b$-functions.
   - Logarithmic comparison problem.
   - Intersection homology D-module.

3. Stratification associated with local $b$-functions.


5. Narvaez’s paper.
Computation of the $b$-functions via embedded resolutions
Resolution of Singularities

Let $f \in \mathcal{O}$ be a convergent power series, $f : \Delta \subseteq \mathbb{C}^n \rightarrow \mathbb{C}$.

Assume that $f(0) = 0$, otherwise $b_{f,0}(s) = 1$.

Let $\varphi : Y \rightarrow \Delta$ be an embedded resolution of $\{f = 0\}$.

If $F = f \circ \varphi$, then $F^{-1}(0)$ is a normal crossing divisor.

**Theorem (Kashiwara).**

There exists an integer $k \geq 0$ such that $b_f(s)$ is a divisor of the product $b_F(s)b_F(s + 1) \cdots b_F(s + k)$. 
Let us consider $f = y^2 - x^3 \in \mathbb{C}\{x, y\}$.

From Kashiwara, the possible roots of $b_f(-s)$ are:

$$1, 1, 1, 2, 5, 7, 4, 3, 5, 11,$$

$$\bar{6}', \bar{3}', \bar{2}', \bar{3}', \bar{6}', 1, \bar{6}', \bar{3}', \bar{2}', \bar{3}', \bar{6}'. $$

Using our algorithm, we have proved that the numbers in red are the roots of $b_f(s)$, all of them with multiplicity one.
Figure: Embedded resolution of $V((xz + y)(x^4 + y^5 + xy^4))$

$$b_f(s) = (s + 1)^2(s + 17/24)(s + 5/4)(s + 11/24)(s + 5/8)$$

$$\quad (s + 31/24)(s + 13/24)(s + 13/12)(s + 7/12)(s + 23/24)$$

$$\quad (s + 5/12)(s + 3/8)(s + 11/12)(s + 9/8)(s + 7/8)$$

$$\quad (s + 19/24)(s + 3/4)(s + 29/24)(s + 25/24)$$
Integral roots of $b$-functions
Let us consider the following example:

\[ A = \begin{pmatrix}
    x_1 & x_2 & x_3 & x_4 \\
    x_5 & x_6 & x_7 & x_8 \\
    x_9 & x_{10} & x_{11} & x_{12}
\end{pmatrix} \]

- \( \Delta_i \) determinant of the minor resulting from deleting the \( i \)-th column of \( A \), \( i = 1, 2, 3, 4 \).
- \( f = \Delta_1 \Delta_2 \Delta_3 \Delta_4 \in \mathbb{C}[x_1, \ldots, x_{12}] \).

From Kashiwara, the possible integral roots of \( b_f(-s) \) are

\[ 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1. \]

Using our algorithm, we have proved that the minimal integral root of \( b_f(s) \) is \(-1\).
Stratification associated with local $b$-functions
Stratrat. associated with local $b$-functions

**Theorem**

- $\text{Ann}_{D[s]}(f^s) + D[s]\langle f \rangle = D[s]\langle \{P_1(s), \ldots, P_k(s), f\} \rangle$

- $l_{\alpha,i} = (l : (s + \alpha)^i) + D[s]\langle s + \alpha \rangle$, $(i = 0, \ldots, m_{\alpha} - 1)$

\[ m_{\alpha}(p) > i \iff p \in V(l_{\alpha,i} \cap \mathbb{C}[x]) \]

1. $V_{\alpha,i} = V(l_{\alpha,i} \cap \mathbb{C}[x])$

2. $\emptyset =: V_{\alpha,m_{\alpha}} \subset V_{\alpha,m_{\alpha}-1} \subset \cdots \subset V_{\alpha,0} \subset V_{\alpha,-1} := \mathbb{C}^n$

3. $m_{\alpha}(p) = i \iff p \in V_{\alpha,i-1} \setminus V_{\alpha,i}$
Experiments

- \( f = (x^2 + 9/4y^2 + z^2 - 1)^3 - x^2z^3 - 9/80y^2z^3 \in \mathbb{C}[x, y, z] \)
- \( b_f(s) = (s + 1)^2(s + 4/3)(s + 5/3)(s + 2/3) \)
- \( V_1 = V(x^2 + 9/4y^2 - 1, z) \)
- \( V_2 = V(x, y, z^2 - 1) \rightarrow \text{two points} \)
- \( V_3 = V(19x^2 + 1, 171y^2 - 80, z) \rightarrow \text{four points} \)
- \( V_3 \subset V_1, \quad V_1 \cap V_3 = \emptyset \)
- \( \text{Sing}(f) = V_1 \cup V_2 \)

- \( \alpha = -1, \quad \emptyset \subset V_1 \subset V(f) \subset \mathbb{C}^3 \)
- \( \alpha = -4/3, \quad \emptyset \subset V_1 \cup V_2 \subset \mathbb{C}^3 \)
- \( \alpha = -5/3, \quad \emptyset \subset V_2 \cup V_3 \subset \mathbb{C}^3 \)
- \( \alpha = -2/3, \quad \emptyset \subset V_1 \subset \mathbb{C}^3 \).
From this, one can easily find a stratification of $\mathbb{C}^3$ into constructible sets such that $b_{f,p}(s)$ is constant on each stratum.

\[
\begin{cases} 
1 & \quad p \in \mathbb{C}^3 \setminus V(f), \\
 s + 1 & \quad p \in V(f) \setminus (V_1 \cup V_2), \\
(s + 1)^2(s + 4/3)(s + 2/3) & \quad p \in V_1 \setminus V_3, \\
(s + 1)^2(s + 4/3)(s + 5/3)(s + 2/3) & \quad p \in V_3, \\
(s + 1)(s + 4/3)(s + 5/3) & \quad p \in V_2.
\end{cases}
\]
Thank you very much!!

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