Invariants via Moving Frames: Computation and Applications

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DART, October 27–30, 2010, Beijing, China
Outline:

- Definitions and examples of various types invariants

- Applications:
  - congruence problem for curves;
  - symmetry reduction of variational problems;

- Structure theorems

- Computation via moving frames (classical, generalized, inductive and algebraic methods)
Group actions and invariants:
Group actions

An action of a group $\mathcal{G}$ on a set $\mathcal{Z}$ is a map $\Phi: \mathcal{G} \times \mathcal{Z} \rightarrow \mathcal{Z}$ such that

i. $\Phi(e, z) = z$, $\forall z \in \mathcal{Z}$.

ii. $\Phi(g_1, \Phi(g_2, z)) = \Phi(g_1 g_2, z)$, $\forall z \in \mathcal{Z}$ and $\forall g_1, g_2 \in \mathcal{G}$.

Example: Let $M(n, \mathbb{K}) = \{n \times n$ matrices over a field $\mathbb{K}\}$.
A group $GL(n, \mathbb{K}) = \{A \in M(n, \mathbb{K}) | \det(A) \neq 0\}$ acts on $\mathbb{K}^n$ by:

$\Phi(A, z) = Az$, $\forall A \in GL(n, \mathbb{K})$ and $z \in \mathbb{K}^n$.

Notation: $\mathcal{G} \rhd \mathcal{Z}$ and $\Phi(g, z) = g \cdot z$. 
We will consider

- $G$ – smooth Lie group or algebraic group over a field $K$
- $Z$ – smooth manifold or algebraic variety
- $\Phi$ – smooth map or polynomial or rational map
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- $G$ – smooth Lie group or algebraic group over a field $K$

- $Z$ – smooth manifold or algebraic variety

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A local action of a topological group $G$ on a topological set $Z$ is a map $\Phi : \Omega \rightarrow Z$ defined on some open subset $\Omega \subset G \times Z$ containing $e \times Z$, such that

i. $\Phi(e, z) = z, \forall z \in Z$.

ii. $\Phi(g_1, \Phi(g_2, z)) = \Phi(g_1 g_2, z), \forall g_1, g_2, z$ such that $(g_2, z) \in \Omega$ and $(g_1 g_2, z) \in \Omega$. 
Invariants:

A function $F$ on $\mathcal{Z}$ is invariant under $\mathcal{G} \bowtie \mathcal{Z}$ if

$$F(g \cdot z) = F(z), \quad \forall z \in \mathcal{Z} \text{ and } \forall g \in \mathcal{G}.$$
Invariants:

A function $F$ on $\mathcal{Z}$ is invariant under $\mathcal{G} \curvearrowright \mathcal{Z}$ if

$$F(g \cdot z) = F(z), \quad \forall z \in \mathcal{Z} \text{ and } \forall g \in \mathcal{G}.\tag{1}$$

A function $F$ on a topological set $\mathcal{Z}$ is a local invariant under $\mathcal{G} \curvearrowright \mathcal{Z}$ if

$$F(g \cdot z) = F(z), \quad \forall(g, z) \in \Omega.\tag{2}$$

for some open subset $\Omega \subset \mathcal{G} \times \mathcal{Z}$ such that $e \times \mathcal{Z} \subset \Omega$.\tag{3}
Invariants under rotations on $\mathbb{R}^2$:

$SO(2, \mathbb{R}) \curvearrowright \mathbb{R}^2$ by rotations

Invariants

- Any smooth invariant on $\mathbb{R}^2 - \{(0, 0)\}$ is functions of $r = \sqrt{x^2 + y^2}$.

- Any polynomial invariant on $\mathbb{R}^2$ is functions of $r^2 = x^2 + y^2$.

Orbits are level sets of $r$. 
Invariants under rotations and translations on $\mathbb{R}^2$:

Action: $SE(2, \mathbb{R}) = SO(2, \mathbb{R}) \times \mathbb{R}^2 \acts \mathbb{R}^2$ by rotations and translations.

$\mathbb{R}^2$ is a single orbit.

Invariants: constant functions.
Differential invariants for planar curves $\gamma(t) = (x(t), y(t))$ under rotations and translations

Action of $SE(2, \mathbb{R})$ on $\mathbb{R}^2$ induces action on $x(t), y(t), \dot{x}(t), \dot{y}(t), \ldots$ (jet bundle of curves in $\mathbb{R}^2$).

- Unit tangent: $T = \left( \frac{dx}{ds}, \frac{dy}{ds} \right)$, $|T| = 1 \Rightarrow$
  
  Infinitesimal arc-length: $ds = \sqrt{\dot{x}^2 + \dot{y}^2} dt$

- Unit normal: $N \perp T$, $|N| = 1$.

- The Frénet equation: $\frac{dT}{ds} = \kappa N$

- Generators of the differential algebra of invariants: $\kappa$ and $\frac{d}{ds}$, where $\frac{d}{ds} = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \frac{d}{dt}$ is an invariant differential operator.

- Fundamental local diff. invariants:
  \[
  \kappa, \kappa_s = \frac{d\kappa}{ds}, \kappa_{ss}, \ldots
  \]
An integral invariants for planar curves \( \gamma(t) = (x(t), y(t)), \ t \in [a, b] \)

Notation: \( X(t) = x(t) - x(a), \ Y(t) = y(t) - y(a), \)

\[
I^{[0,1]}(t) = \int_{a}^{t} Y(\tau) \, dX(\tau) - \frac{1}{2} X(\tau) Y(\tau)
\]

\( I^{[0,1]} \) is invariant under \( SA(2, \mathbb{R}) \supset SE(2, \mathbb{R}) \)
An **discrete** invariants for quadratic forms

The standard action of $GL(n, \mathbb{C})$ on $\mathbb{C}^n$ induces an action on the space $V^n_d$ of homogeneous polynomials of degree $d$ in $n$ variables:

$$A \cdot P(x) = P(A^{-1}x), \, \forall A \in GL(n, \mathbb{C}) \text{ and } x \in \mathbb{C}^n.$$

There are well known canonical forms for $GL(n, \mathbb{C}) \curvearrowright V^n_2$:

$$x_1^2 + \cdots + x_k^2, \, \text{for } k = 0, \ldots n.$$

$k$ is a discrete invariant for $GL(n, \mathbb{C}) \curvearrowright V^n_2$. 
Types of the invariants:

- local smooth;
- polynomial, rational, and algebraic;
- differential;
- integral;
- integro-differential;
- discrete;
- ...
Applications:

• Equivalence (congruence) problems for
  – sub-manifolds (in particular curves and surfaces)
  – for polynomials
  – differential equations
  – ...

• Symmetry reduction of
  – differential equations
  – variational problems
  – algebraic equations

• Invariant geometric flows
  • ...
  •...
Equivalence problem for curves
Equivalence problem for curves in $\mathbb{R}^n$.

- **Problem:** Given an action of a group $G$ on $\mathbb{R}^n$ and curves $\gamma_1 : [a, b] \rightarrow \mathbb{R}^n$ and $\gamma_2 : [c, d] \rightarrow \mathbb{R}^n$ decide whether there exists $g \in G$ such that

  $$\text{Image}(\gamma_1) = g \cdot \text{Image}(\gamma_2).$$

- If such $g \in G$ exists then $\gamma_1$ and $\gamma_2$ are called $G$-equivalent, or $G$-congruent:

  $$\gamma_1 \cong \gamma_2.$$
Transformations on $\mathbb{R}^2$ commonly appearing in computer image processing:

- **Special Euclidean** (orientation preserving rigid motions):
  \[ X = \cos(\phi)x - \sin(\phi)y + a, \quad Y = \sin(\phi)x + \cos(\phi)y + b. \]

- **Euclidean** (rigid motions):
  \[ X = \cos(\phi)x - \sin(\phi)y + a, \quad Y = \epsilon(\sin(\phi)x + \cos(\phi)y) + b \]
  \[ \epsilon = \pm 1 \]

- **Similarity**
  \[ X = \lambda(\cos(\phi)x - \sin(\phi)y) + a, \quad Y = \epsilon\lambda(\sin(\phi)x + \cos(\phi)y) + b, \]
  \[ \epsilon = \pm 1, \lambda \neq 0. \]

- **Equi-affine** (area and orientation preserving):
  \[ X = \alpha x + \beta y + a, \quad Y = \gamma x + \delta y + b, \]
  \[ \alpha\delta - \beta\gamma = 1 \]

- **Affine**:
  \[ X = \alpha x + \beta y + a, \quad Y = \gamma x + \delta y + b \]
  \[ \alpha\delta - \beta\gamma \neq 0 \]

- **Projective**:
  \[ X = \frac{\alpha x + \beta y + a}{\nu x + \mu y + c}, \quad Y = \frac{\gamma x + \delta y + b}{\nu x + \mu y + c}, \quad \det \begin{pmatrix} \alpha & \beta & a \\ \gamma & \delta & b \\ \nu & \mu & c \end{pmatrix} \neq 0 \]
Euclidean and equi-affine frame

Euclidean geometry in $\mathbb{R}^2$

$$SE(2, \mathbb{R}) = SO(2, \mathbb{R}) \times \mathbb{R}^2$$

Equi-affine geometry in $\mathbb{R}^2$

$$SA(2, \mathbb{R}) = SL(2, \mathbb{R}) \times \mathbb{R}^2$$

Moving Frame:

$$T = \left( \frac{dx}{ds}, \frac{dy}{ds} \right), \quad N \perp T, \quad |N| = 1$$

$$T = \left( \frac{dx}{d\alpha}, \frac{dy}{d\alpha} \right), \quad N = \frac{dT}{d\alpha}$$

Infinitesimal arc-length:

$$|T| = 1 \Rightarrow ds = \sqrt{1 + y_x^2} \, dx$$

$$\det |TN| = 1 \Rightarrow d\alpha = \frac{1}{3} y_{xx} \, dx$$

Fundamental differential invariants:

$$\frac{dT}{ds} = \kappa N$$

$$\kappa_s = \frac{d\kappa}{ds}, \kappa_{ss}, \ldots$$

$$\frac{dN}{d\alpha} = \mu T$$

$$\mu_\alpha = \frac{d\mu}{d\alpha}, \mu_{\alpha\alpha}, \ldots$$
Differential invariants for planar curves

Let $\mathcal{G}$ be an $r$-dim’l Lie group acting on the plane. For almost all actions $\exists$

- a local differential invariant $\xi$ ($\mathcal{G}$-curvature) of differential order $r - 1$;

- an invariants differential form $\omega$ (infinitesimal $\mathcal{G}$-arclength) of differential order at most $r - 2$ and the dual invariant differential operator $D_\omega$.

such that any other differential invariant is a smooth function of $\xi, D_\omega \xi, D^2_\omega \xi, \ldots$
Relations between invariants of a group and its subgroup

• Euclidean: $\kappa = \frac{(y \dot{x} - \dot{y} \dot{x})}{(x^2 + y^2)^{3/2}}$, $ds = \sqrt{x'^2 + y'^2} \, dt$, $\frac{d}{ds} = \frac{1}{\sqrt{x'^2 + y'^2}} \frac{d}{dt}$

• equi-affine: $\mu = \frac{3 \kappa (\kappa_{ss} + 3 \kappa^3) - 5 \kappa^2_s}{9 \kappa^{8/3}}$, $d\alpha = \kappa^{1/3} ds$, $\frac{d}{d\alpha} = \frac{1}{\kappa^{1/3}} \frac{d}{ds}$

• projective: $\eta = \frac{6 \mu_{\alpha \alpha \alpha} \mu_\alpha - 7 \mu_{\alpha \alpha}^2 - 9 \mu_{\alpha \alpha} \kappa}{6 \mu_\alpha^{8/3}}$, $d\rho = \mu^{1/3} d\alpha$, $\frac{d}{d\rho} = \frac{1}{\mu^{1/3}} \frac{d}{d\alpha}$.

Definition: Curves for which $\mathcal{G}$-curvature or $\mathcal{G}$-arclength are undefined are called $\mathcal{G}$-exceptional.

*see (Kogan 2001, 2003) for a general method of deriving invariants of a group in terms of invariants of its subgroup
Congruence criteria for curves with specified initial point

• Theorem: Two non $G$-exceptional curves are $G$-congruent iff their $G$-curvatures as functions of $G$-arclength coincide.

For $\gamma_1(t)$, $t \in [a, b] \rightarrow \mathbb{R}^2$ and $\gamma_2(\tau)$, $\tau \in [c, d] \rightarrow \mathbb{R}^2$:

$$\exists g \in G \; \text{s. t.} \; g \cdot \gamma_1(a) = \gamma_2(c) \quad \text{and} \quad \text{Image}(\gamma_1) = g \cdot \text{Image}(\gamma_2)$$

$\Updownarrow$

$$\xi|_{\gamma_1}(s_1) = \xi|_{\gamma_2}(s_2), \; \text{where} \; s_1(t) = \int_a^t \varpi|_{\gamma_1} \quad \text{and} \quad s_2(\tau) = \int_c^\tau \varpi|_{\gamma_2}$$

• Applicable only if:
  – initial point is specified
  – arc-length reparametrization is feasible in practice
$\mathcal{G}$-curvature under reparametrization

Euclidean example: $\kappa = \frac{(\dot{y}\ddot{x} - \dot{x}\ddot{y})}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$

$\gamma(t) = (t, \cos t), \ t \in [0, \pi]$ \quad $\tilde{\gamma}(\tau) = (\sqrt{\tau}, \cos \sqrt{\tau}), \ \tau \in [0, \pi^2]$

$\kappa|_{\gamma(\phi(\tau))} = -\frac{\cos(t)}{(1+\sin^2(t))^{3/2}}$ \quad $\kappa|_{\tilde{\gamma}(\tau)} = -\frac{\cos(\sqrt{\tau})}{(1+\sin^2(\sqrt{\tau}))^{3/2}}$

$\kappa|_{\gamma(\phi(\tau))} = \kappa|_{\tilde{\gamma}(\tau)}$ where $t = \phi(\tau) = \sqrt{\tau}$. 
Differential signature for planar curves

(Calabi et al. (1998))

• Let $\xi$ be $G$-curvature, $\varpi$-infinitesimal $G$-arclength and $\xi\varpi = D\varpi\xi$.

• **Definition:** The $G$-signature of a non-exceptional curve $\gamma(t) = (x(t), y(t))$, $t \in [a, b]$ is the image of a parametric curve $((\xi|_{\gamma(t)}, \xi\varpi|_{\gamma(t)}))$:

$$S_{\gamma}(t) = \{(\xi|_{\gamma(t)}, \xi\varpi|_{\gamma(t)}) \mid t \in [a, b]\}.$$ 

• **$G$-congruence criterion for non-exceptional curves**

$$\gamma_1 \cong \gamma_2 \quad \downarrow \quad \uparrow \quad \text{under certain conditions}$$

$$S_{\gamma_1} = S_{\gamma_2}$$
Example 1 of Euclidean differential signature:

<table>
<thead>
<tr>
<th>Curve $\gamma(t)$</th>
<th>Curve $\tilde{\gamma}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma(t) = (\sqrt{t}, \cos \sqrt{t})$, $t \in [0, \pi^2]$</td>
<td>$\tilde{\gamma}(t) = (\frac{3}{5}t - \frac{4}{5} \sin t, \frac{4}{5}t + \frac{3}{5} \sin t)$, $t \in [0, \pi]$</td>
</tr>
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Images of $\gamma$ and $\tilde{\gamma}$ in $\mathbb{R}^2$

Signatures $(\kappa^2, \kappa_s^2)$ for $\gamma$ and $\tilde{\gamma}$
Example 2 of Euclidean differential signature:

\[ \gamma(t) = (t, \cos t), \ t \in [0, \pi] \quad \tilde{\gamma}(t) = (t, \cos t), \ t \in [0, 2\pi] \]

Images of \( \gamma \) and \( \tilde{\gamma} \) in \( \mathbb{R}^2 \)

Images of signatures of \( \gamma \) and \( \tilde{\gamma} \) coincide due to reflection symmetry of \( \tilde{\gamma} \)

Signature for \( \gamma \) is traced 2 times when \( t \in [0, \pi] \) due to symmetry under rotations by \( \pi \) around the point \( (\frac{\pi}{2}, 0) \).
Signature for \( \tilde{\gamma} \) is traced 4 times when \( t \in [0, 2\pi] \)!
Local $G$-congruence criterion for non-exceptional curves

$\gamma_1$ locally congruent to $\gamma_2$

$\downarrow \uparrow$ for smooth curves

$S_{\gamma_1}$ and $S_{\gamma_2}$ overlap
Advantages and disadvantages of differential signature

+ the construction extends to curves and higher dimensional submanifolds of $\mathbb{R}^n$ under majority of transformations.

+ independent of parametrization

+ can be used for local comparison

+ can be used to detect symmetries

- depends on derivatives of high order (for planar curves of order $= \dim G$) $\implies$ very sensitive to high frequency perturbations
Sensitivity of differential signature to high frequency perturbation:

Images of $\gamma = (t, \cos t)$ and $\tilde{\gamma} = (t, \cos(t) + \frac{1}{100} \sin(100t))$, $t \in [0, \pi]$
Integral variables for planar curves $\gamma(t) = (x(t), y(t))$.

(Hann and Hickman (2002)

- $G$-action on $\mathbb{R}^2$ induces an action on $x(0)$, $y(0)$, $x(t)$, $y(t)$, and
  $$x[i,j](t) = \int_0^t x(\tau)^i y(\tau)^j \, dx(\tau).$$

- Example: if $x \to x + y$, and $y \to y$ then
  $$x[i,j] \to \int_0^t [x(\tau) + y(\tau)]^i y(\tau)^j \, d [x(\tau) + y(\tau)]$$

- $y[i,j](t) = \int_0^t x(\tau)^i y(\tau)^j \, dy(\tau)$ can be expressed in terms of $x(0), y(0), x(t), y(t), x[k,l](t) = \int_0^t x(\tau)^k y(\tau)^l \, dx(\tau)$ via integration-by-parts.

- $i + j$ the order of integral variable $x[i,j]$.  

27
Integral invariants for planar curves

- An affine action can be prolonged to an integral jet bundle of planar curves which is parametrized by $x(0), y(0), x, y, x^{[i,j]}$, where $j > 0, i \geq 0$.

- Integral invariants are invariant function on integral jet bundle.

- Moving frame method can be applied to derive fundamental or generating sets of integral invariants.

- In (Feng, Kogan, Krim (2010)) we derived Euclidean and affine integral fundamental sets of invariants for curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ via inductive variation of the moving frame method.

*Integral invariants defined here are not the same as moment invariants (Taubin and Cooper (1992))
Examples of integral invariants for planar curves

\[ \gamma(t), \quad t \in [a, b] \]

- Notation: \( X(t) = x(t) - x(a), \) \( Y(t) = y(t) - y(a), \)

\[ X[i,j](t) = \int_a^t X(\tau)^i Y(\tau)^j dX(\tau). \]

- Invariants:

0-th order \( r = \sqrt{X^2 + Y^2} - E_2\)-invariant

1-st order \( I^{[0,1]} = X^{[0,1]} - \frac{1}{2}XY - (SA_2 \supset SE_2)\)-invariant.

2-nd order \( \ast I^{[1,1]} = YX^{[1,1]} - \frac{1}{2}XX^{[0,2]} - \frac{1}{6}X^2Y^2. \)

\( SA_2 \) and \( E_2\)-invariant

\( \ast I^{[0,2]} = YX^{[0,2]} + 2XX^{[1,1]} - \frac{1}{3}XY^3 - \frac{2}{3}X^3Y \)

\( E_2\)-invariant

\[ \ldots \]
Geometric interpretation of

\[ I_{[0,1]} = X_{[0,1]} - \frac{1}{2} X Y = \int_{0}^{t} Y \, dX - \frac{1}{2} X Y \]
Examples of integral signatures for planar curves

- $SE(2)$-signature $(r, I^{[0,1]})$
- $E(2)$- signatures $(r, (I^{[0,1]})^2)$ or $(r, I^{[1,1]})$.
- Similarity signature: $\left( \frac{(I^{[0,1]})^2}{r^4}, \frac{I^{[1,1]}}{r^4} \right)$
- $SA(2)$-signature $(I^{[0,1]}, I^{[1,1]})$
Reasonable behavior under high frequency perturbation:

\[ \gamma(t) = (t, \cos t), \quad t \in [0, \pi] \]
\[ \tilde{\gamma}(t) = (t, \cos t + \frac{1}{100} \sin(100t)), \quad t \in [0, \pi] \]

Images of \( \gamma \) and \( \tilde{\gamma} \) in \( \mathbb{R}^2 \)

\( SE(2, \mathbb{R}) \)- signatures \((r, I_1)\) for \( \gamma \) and \( \tilde{\gamma} \)
Signature \((r, I_1)\) for \(\gamma(t) = (t, \cos(t))\) for \(t \in [0, 6\pi]\):
Equivalence theorem for curves with specified initial points:

\[ \gamma_1 \sim \gamma_2 \]

\[ \Downarrow \quad \Uparrow \quad \text{conditions?} \]

\[ \text{integral signature}|_{\gamma_1} = \text{integral signature}|_{\gamma_2} \]

Remark:

- \( \Downarrow \) follows from the definition of invariants

- \( \Uparrow \) is proved for

  - \( SE(2, \mathbb{R}) \)-signature \( (r, I^{[0,1]}) \)

  - \( E(2, \mathbb{R}) \)-signature \( (r, I^{[1,1]}) \).
Advantages and disadvantages of integral signature

+ extends to curves in $\mathbb{R}^n$ (see Feng, Kogan, Krim (2010) for curves in $\mathbb{R}^3$).

+ independent of parametrization

+ tolerant to data uncertainty and perturbations

∓ requires an identified initial point

- possible, but problematic use for local comparison

- no straightforward generalization to rational action (i.e. projective actions), see Hann and Hickman (2002) for a numeric approach.)
equivalence problems

General framework for solving an equivalence problem for an action of $G$ on a set $\mathcal{Z}$

• find a finite set of invariants that separates generic orbits. i.e. orbits on an open dense subset $\mathcal{U} \subset \mathcal{Z}$.

• characterize orbits on $\mathcal{U} – \mathcal{Z}$ (possibly by another set of invariants).
A glimpse into the symmetry reduction
General framework for symmetry reduction

**Definition:** A group of transformations $\mathcal{G}$ on the space of independent and dependent variables is a Lie symmetry of a differential equation (or a variational problem) if each element of $\mathcal{G}$ maps a solution to a solution.

**Theorem:** (S. Lie (1897))

- (almost) any $\mathcal{G}$-symmetric system of differential equations can be written in terms of differential $\mathcal{G}$-invariants.

- (almost) any $\mathcal{G}$-symmetric variational problem can be written in terms of differential $\mathcal{G}$-invariants and $\mathcal{G}$-invariant differential forms.
Example: $SE(2, \mathbb{R})$-invariant variational problem for $y = u(x)$:

\[
\mathcal{L}[u] = \int \frac{u_{xx}^2}{2(1+u_x^2)^{5/2}} dx \quad \iff \quad \mathcal{L}[\kappa] = \int \frac{1}{2} \kappa^2 ds
\]

\[
E = \frac{\partial}{\partial u} - \left( \frac{d}{dx} \right) \frac{\partial}{\partial u_x} + \left( \frac{d}{dx} \right)^2 \frac{\partial}{\partial u_{xxx}} \ldots
\]

\[
\Delta = 0 \quad \iff \quad \kappa_{ss} + \frac{1}{2} \kappa^3 = 0
\]

\[
\Delta = \frac{2 u_4 (1 + u_1^2)^2 - 20 u_1 u_2 u_3 (1 + u_1^2) + 30 u_2^3 u_1^2 - 5 u_2^3}{2 (1 + u_1^2)^{9/2}}.
\]

($u_1 = u_x, \ldots, u_4 = u_{xxxx}$)
**G-invariant Euler-Lagrange operator for curves** $y = u(x)$:

\[
\int L(x, u, u_1, \ldots, u_n) \, dx \quad \Leftrightarrow \quad \int L(\xi, D\omega\xi, \ldots, D^m\omega\xi) \, \omega
\]

\[
E(L) = \sum_{i} \left(-\frac{d}{dx}\right)^i \frac{\partial L}{\partial u_i} = 0 \quad \Leftrightarrow \quad [A^*E(L) - B^*H(L)] = 0,
\]

where

\[
E(L) = \sum_{i=0}^{n} (-D\omega)^i \frac{\partial L}{\partial \xi_i}, \quad H(L) = \sum_{i>j \geq 0} \xi_{i-j} (-D\omega)^j \frac{\partial L}{\partial \xi_i} - L.
\]

- $A^*$ and $B^*$ – $G$-invariant diff. operators, computable by differentiation and linear algebra.
- **general formula for any number of independent variables and unknown functions is obtained in Kogan and Olver(2003)**
- Completely algorithmic – iVB package (IK) in **Maple**.

40
Structure theorems
Structure theorems of algebraic invariant theory:

- Hilbert theorem (1890): If an algebraic reductive group $G$ acts regularly on an affine variety $Z$ then the ring of polynomial invariants $K[Z]^G$ is finitely generated.

$$K[Z]^G = K[u_1, \ldots, u_d] \setminus R,$$

where $R$ is a finitely generated ideal of syzygies.

- Rosenlicht theorem (1956): If an algebraic group $G$ acts rationally on an affine variety $Z$ of dimension $m$ then the field of rational invariants $K(Z)^G$ is finitely generated.

If $\dim Z = m$ and $\max_z \dim O_z = r$, then the transcendence degree of $K(Z)^G : K$ is $m - r$. 
• Problems

– Find (minimal) generating set of $\mathbb{K}[\mathbb{Z}]^G$ and $\mathbb{K}(\mathbb{Z})^G$.

– Describe the structure of $\mathbb{K}[\mathbb{Z}]^G$ and $\mathbb{K}(\mathbb{Z})^G$ (find syzygy ideal, transcendence basis, ...).
Theorem of smooth invariant theory:

- **Definition**: Let $G$ be a smooth Lie group acting on a smooth manifold $\mathcal{Z}$. A collection of local invariants on an open subset $U \subset \mathcal{Z}$ forms a fundamental set if they are functionally independent, and any local invariant on $U$ can be expressed as a smooth function of the invariants from this set.

- **Frobenious integrability theorem**: If $\dim \mathcal{Z} = m$ and all orbits have the same dimension $r$, then for each point $z \in \mathcal{Z}$ there exists a fundamental set of $m - r$ local smooth invariants defined on an open neighborhood $U_z$. 
Structure theorem of differential invariant theory:

Let $G$ be a Lie group acting on an $m$-dim’l manifold $Z$. For $1 \leq p < m$ ∃!prolongation of $G$-action to the jet bundle $J(Z, p)$ of $p$-dim’l sub-manifolds of $Z$.

Tresse theorem (1894): Local smooth invariants on $J(Z, p)$ have a structure of finitely generated differential algebra*:

- $\exists \{I^1, \ldots, I^\nu\}$ - invariant function on $J(Z, p)$
- $\exists D_1, \ldots, D_p$ - invariant differential operators

such that any invariant $I$ on $J(Z, p)$ can be expressed as

$$I = F \left( \ldots, D_J(I^l), \ldots \right)$$

*in general it is a non-free algebra with non-commutative derivations
Problem:

- Find (minimal) set of generators

- Finite (minimal) set of generating syzygies $H \left( \ldots, D_J(I^l), \ldots \right) \equiv 0$
Structure theorems of integral invariant theory ???

or may be

Structure theorems of integro-differential invariant theory ???
Invariants via moving frames

- **Classical moving frames** (Frénet (1847), Serret (1851), Darboux (1887), Cartan (1935))

- **Generalization of moving frame construction to arbitrary Lie group actions on manifolds** (Fels and Olver (1999))

- **Inductive and recursive variations** (Kogan(2001, 2003))

- **Algebraic formulation** (Hubert, Kogan(2007))
Euclidean and affine moving frames for curves

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<th>Equi-affine geometry in $\mathbb{R}^2$</th>
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<td>$SE(2) = SO(2) \rtimes \mathbb{R}^2$</td>
<td>$SA(2) = SL(2) \rtimes \mathbb{R}^2$</td>
</tr>
</tbody>
</table>

**Moving Frame:**

- Euclidean: $T = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$, $N \perp T$, $|N| = 1$
- Equi-affine: $T = \left(\frac{dx}{d\alpha}, \frac{dy}{d\alpha}\right)$, $N = \frac{dT}{d\alpha}$

**Infinitesimal arc-length:**

- $|T| = 1 \Rightarrow ds = \sqrt{1 + y_x^2} \, dx$
- $\det |TN| = 1 \Rightarrow d\alpha = \frac{1}{3} y_{xx} \, dx$

**Fundamental differential invariants:**

- Euclidean: $\frac{dT}{ds} = \kappa N$, $\kappa_s = \frac{d\kappa}{ds}$, $\kappa_{ss}$, ...
- Equi-affine: $\frac{dN}{d\alpha} = \mu T$, $\mu_\alpha = \frac{d\mu}{d\alpha}$, $\mu_{\alpha\alpha}$, ...
Observe that in affine and Euclidean case:

- Moving frame defines a map from the jets of curve to $\mathcal{G}$, i.e. $([T, N], (x, y)) \in \mathcal{G}$.

- Invariants can be obtained from the pull-backs of a basis of invariant differential forms on $\mathcal{G}$ by $\rho$


Definition. (Fels and Olver (1999)) Given $\mathcal{G} \curvearrowright \mathcal{Z}$, a (local) moving frame is an equivariant smooth (local) map $\rho : \mathcal{Z} \to \mathcal{G}$.
**Theorem.** (Fels and Olver (1999)) \( \exists \) loc. moving frame \( \iff \) local cross-section

\( \mathcal{K} \) on \( \mathcal{Z} \), s.t. \( \text{codim} \mathcal{K} = \dim \mathcal{G} \):

\( \rho : \mathcal{Z} \to \mathcal{G} \) is defined by the condition \( \rho(z) \cdot z \in \mathcal{K} \)

\[
\rho(g \cdot z)(g \cdot z) = \rho(z) \cdot z, \text{ freeness } \implies \rho(g \cdot z) = \rho(z)g^{-1}
\]

\( \downarrow \)

\( \rho \) is a \( \mathcal{G} \)-equivariant map.
Implicit invariantization $\iota$: 

Let $z^1, \ldots, z^d$ be loc. coordinates on $\mathcal{Z}$ and $\mathcal{K}$ be a loc. cross-section.

- **functions:**
  \[ \forall f \in \mathcal{F}(\mathcal{Z}) \quad \exists! \text{ loc. inv. } \iota f \in \mathcal{F}(\mathcal{Z}) \text{ s. t. } \iota f|_\mathcal{K} = f|_\mathcal{K}. \]
  \[ \{\iota(z^1), \ldots, \iota(z^n)\} \supset \text{ fundamental set of inv.} \]

- **vector fields:**
  \[ \forall V \in T^*_\mathcal{Z} \quad \exists! \text{ loc. inv. } \iota V \text{ s. t. } \iota V|_\mathcal{K} = V|_\mathcal{K}. \]
  \[ D_1 = \iota \left( \frac{d}{dz^1} \right), \ldots, D_n = \iota \left( \frac{d}{dz^n} \right) \text{ is a basis of invariant differential operators (non-commutative in general)} \]

- **differential forms:**
  \[ \forall \Omega \in \Lambda^k \quad \exists! \text{ loc. inv. } \iota \Omega \text{ s. t. } \iota \Omega|_\mathcal{K} = \Omega|_\mathcal{K}. \]
  \[ \varpi = \iota dz^1, \ldots, \varpi_n = \iota dz^n \text{ is the dual basis of invariant differential 1-forms} \]
Explicit invariantization steps:

1. Write down a system of equations that describes $g \in \mathcal{G}$ which brings an arbitrary point $z \in \mathcal{Z}$ to the cross-section;

2. Solve the system for the group parameters ($g = \rho(z)$);

3. Substitute $\rho(z)$ with $g$ in the pull-back of a function (or a form) by the action of $g \in \mathcal{G}$. 
Explicit invariantization steps:

1. Write down a system of equations that describes \( g \in \mathcal{G} \) which brings an arbitrary point \( z \in \mathcal{Z} \) to the cross-section;

2. Solve the system for the group parameters \( (g = \rho(z)) \);

3. Substitute \( \rho(z) \) with \( g \) in the pull-back of a function (or a form) by the action of \( g \in \mathcal{G} \).

Constructive idea in the algebraic setting is to replace steps 2 and 3 with elimination of the group parameters (Hubert, Kogan (2007)).
Example: $SO(\mathbb{R}, 2) \sim \mathbb{R}^2 - \{(0, 0)\}$:

**Action:**

\[
X = \cos(\phi)x - \sin(\phi)y, \\
Y = \sin(\phi)x + \cos(\phi)y.
\]

**Cross-section:**

\[\mathcal{K} = \{(x, y) | x = 0, y > 0\}\]

1. Equations: $\cos(\phi)x - \sin(\phi)y = 0, \quad Y = \sin(\phi)x + \cos(\phi)y > 0$.

2. Solution: $\cos \phi = \frac{y}{\sqrt{x^2 + y^2}}, \sin \phi = \frac{x}{\sqrt{x^2 + y^2}}$

3. Substitution:
   - into $Y \Rightarrow r = \sqrt{x^2 + y^2}$ - invariant function;
   - into $dX \Rightarrow \varpi_1 = \frac{1}{\sqrt{x^2 + y^2}}(y \, dx - x \, dy)$
   - into $dY \Rightarrow \varpi_2 = \frac{1}{\sqrt{x^2 + y^2}}(x \, dx + y \, dy)$
\[ SE(2, \mathbb{R}) = SO(2, \mathbb{R}) \times \mathbb{R}^2 \sim \text{on plane curves:} \]

\[ X = \cos(\phi)x - \sin(\phi)y + a, \quad Y = \sin(\phi)x + \cos(\phi)y + b \]

\[ Y_X = \frac{\sin(\phi) + \cos(\phi)y_x}{\cos(\phi) - \sin(\phi)y_x}, \quad Y_{XX} = \frac{y_{xx}}{(\cos(\phi) - \sin(\phi)y_x)^3}, \]

\[ Y_{XXX} = \frac{(\cos(\phi) - \sin(\phi)y_x)y_{xxx} + 3\sin(\phi)y_x^2}{(\cos(\phi) - \sin(\phi)y_x)^5}. \]

\[ \text{cross-section: } \mathcal{K} = \{x = 0, y = 0, y_x = 0\} \]

\[ \downarrow \]

solve \(X = 0, Y = 0, Y_X = 0\) : for \(a, b, \phi \Rightarrow \text{moving frame:} \]

\[ \cos \phi = \frac{1}{\sqrt{y_x^2 + 1}}, \quad \sin \phi = -\frac{y_x}{\sqrt{y_x^2 + 1}}, \quad a = -\frac{x + y_x y}{\sqrt{y_x^2 + 1}}, \quad b = \frac{y_{xx} - y}{\sqrt{y_x^2 + 1}}. \]
Substitute: \( \cos \phi = \frac{1}{\sqrt{y_x^2+1}} \), \( \sin \phi = -\frac{y_x}{\sqrt{y_x^2+1}} \) into

\[
Y_{XX} = \frac{y_{xx}}{(\cos(\phi) - \sin(\phi)y_x)^3} \quad \Rightarrow \quad I_2 = \kappa = \frac{y_{xx}}{(1+y_x^2)^{3/2}}
\]

\[
Y_{XXX} \quad \Rightarrow \quad I_3 = \kappa_s = \frac{y_{xxx}(1+y_x^2)-3y_x y_{xx}^2}{(1+y_x^2)^{5/2}}
\]

\[
Y_{XXXX} \quad \Rightarrow \quad I_4 = \kappa_{ss} + 3 \kappa^3
\]

\[
dX = \cos(\phi)dx - \sin(\phi)dy \quad \Rightarrow \quad \varpi = \frac{dx+y_xdy}{\sqrt{1+y_x^2}} = \sqrt{1+y_x^2} \; dx + \frac{y_x}{\sqrt{1+y_x^2}} \; \theta,
\]

where \( \theta = dy - y_x \; dx \).
Recursive and inductive variations of a moving frame construction.

(Kogan 2000, 2003)

- **Recursive:**
  - does not require freeness, but requires a slice - a cross-section with a constant isotropy group;
  - on a jet bundle allows to construct moving frames and invariants order-by-order.

- **Inductive:**
  - requires splitting of the group into a product of two subgroups $G = AB$ s. t. $A \cap B$ is discrete;
  - invariants and moving frames for $A$ (or $B$) can be used to construct invariants and a moving frame for $G$.

\[\downarrow\]

Relations among the invariants of $G$ and its subgroups.
Ex.: from the Euclidean to the affine action on the planar curves.

\[ SA(2, \mathbb{R}) = SL(2, \mathbb{R}) \times \mathbb{R}^2 = B \cdot A, \text{ where } A = SE(2, \mathbb{R}) \text{ and } \]

\[ B = \left\{ \begin{pmatrix} \tau & \lambda \\ 0 & \frac{1}{\tau} \end{pmatrix} \right\} \]

Notation: \( y_1 = y_x, \ y_2 = y_{xx}, \ldots \)

\( \mathcal{K}_A = \{ z \in \mathcal{J}^k \mid x = 0, y = 0, y_1 = 0 \} \) is stable under the \( B \)-action.

\( \mathcal{K}_B = \{ z \in \mathcal{K}_A \mid y_2 = 0, y_3 = 1 \} \subset \mathcal{K}_A \) is a cross-section to the \( SA(2, \mathbb{R}) \)-action on the jets of curves.

\[ \Downarrow \]

a moving frame for \( B \) on \( \mathcal{K}_A^4 \)

\[ \Downarrow \]

\[ \mu = \frac{\kappa(\kappa_{ss} + 3\kappa^3) - \frac{5}{3}\kappa_s^2}{\kappa^{8/3}}, \quad d\alpha = \kappa^{1/3}ds, \quad \frac{d}{d\alpha} = \frac{1}{\kappa^{1/3}} \frac{d}{ds} \]
Example: from the affine to the projective action on the planar curves.

\[ PGL(3, \mathbb{R}) = B \cdot A, \] where \( A = SL(2, \mathbb{R}) \) and \( B = \left\{ \begin{pmatrix} 1 & ab & 0 \\ 0 & a & 0 \\ b & c & 1/a \end{pmatrix} \right\} \).

\[ \mathcal{K}_A = \{ z \in J^k | x = 0, y = 0, y_x = 0, y_{xx} = 1, y_{xxx} = 0 \} \] is stable under the \( B \)-action.

\[ \mathcal{K}_B = \{ z \in \mathcal{K}_A | y_4 = 0, y_5 = 1, y_6 = 0 \} \subset \mathcal{K}_A \] is a cross-section to the \( PGL(3, \mathbb{R}) \)-action on the jets of curves.

\[ \downarrow \]

moving frame for \( B \) on \( \mathcal{K}_A \)

\[ \downarrow \]

\[ \eta = \frac{-7\mu_{\alpha\alpha}^2 + 6\mu_{\alpha}\mu_{\alpha\alpha\alpha} - 3\mu_{\alpha}^2}{6\mu_{\alpha}^{8/3}}, \quad d\varrho = \mu_{\alpha}^{1/3} d\alpha, \quad \frac{d}{d\varrho} = \frac{1}{\mu_{\alpha}^{1/3}} \frac{d}{d\alpha} \]
Algebraic formulation of the moving frame method.

(Hubert, Kogan 2007)

- applicable to rational actions of algebraic groups
- replaces non-constructive step of solving for group parameters with constructive elimination algorithms
- produces a generating set of rational invariants
- produces a set of algebraic invariants with replacement property, (corresponds to invariantization of coordinate functions in the smooth construction).
Ideals and varieties

\( G \curvearrowright \mathcal{Z} = \mathbb{K}^n, \text{char}\mathbb{K} = 0. \)

- **source and target space:** \( \mathcal{Z} \times \mathcal{Z} = \mathbb{K}^n \times \mathbb{K}^n, \)
  \( \mathbb{K}[z, Z] = \mathbb{K}[z_1, \ldots, z_n, Z_1, \ldots, Z_n], \)

- **group:** \( G = \mathcal{V}(G) \subset \mathbb{K}^l, G \subset \mathbb{K}[\lambda] = \mathbb{K}[\lambda_1, \ldots, \lambda_l] \text{ radical} \)

- **action:** \( A = (Z_1 - g_1(\lambda, z), \ldots, Z_n - g_n(\lambda, z)) \subset h^{-1}\mathbb{K}[\lambda, z, Z] \)
  \( J = G + A \subset h^{-1}\mathbb{K}[\lambda, z, Z] \text{ radical} \)

- **graph:** \( O = J \cap \mathbb{K}[z, Z] \Rightarrow O^e \subset \mathbb{K}(z)[Z] \text{ radical} \)

- \( \exists \ K \subset \mathbb{K}[Z] \text{ prime such that} \)
  \( I^e = (O^e + K) \subset \mathbb{K}(z)[Z] \text{ radical, zero-dimensional} \)

- **cross-section:** \( \mathcal{K} = \mathcal{V}(K) \subset \mathbb{K}^n \)
Example: $SO(2, \mathbb{R}) \sim \mathbb{R}^2$.

- **group:** $G = (\lambda_1^2 + \lambda_2^2 - 1) \subset \mathbb{R}[\lambda_1, \lambda_2], \quad (\lambda_1 = \cos \phi, \lambda_2 = \sin \phi)$
- **action:** $J = A + G$, where
  \[
  A = (Z_1 - \lambda_1 z_1 - \lambda_2 z_2, \quad Z_2 - \lambda_2 z_1 + \lambda_1 z_2)
  \]
- **graph:** $O = J \cap \mathbb{R}[z, Z] = \langle Z_1^2 + Z_2^2 - z_1^2 - z_2^2 \rangle$.
  \[
  O^e = \langle Z_1^2 + Z_2^2 - (z_1^2 + z_2^2) \rangle \subset \mathbb{R}(z)[Z].
  \]
- **cross-section:** $K = (Z_1)$
- $I^e = O^e + K = \langle Z_1, Z_2^2 - (z_1^2 + z_2^2) \rangle$
- $\mathbb{R}(Z)^G = \mathbb{R}(z_1^2 + z_2^2)$
- **2 replacement invariants:** $\xi^{(\pm)} = \pm \sqrt{z_1^2 + z_2^2}$. 
\[ I^e = (O^e + K) \subset \mathbb{R}(z)[Z] \] radical, zero-dimensional.

**Properties:**

- Coefficients of a reduced Gröbner basis of \( I^e \) generate \( \mathbb{R}(z)^G \) (the same is true for \( O^e \)).

- \( I^G = I^e \cap \mathbb{R}(z)^G[Z] = \langle Q \rangle \) is prime.

- If c.-s. \( \mathcal{K} \) intersects generic orbit at \( d \) points then \( I^G \) has \( d \) zeros of \( n \)-tuples \( \xi^{(i)} = (\xi_1^{(i)}, \ldots, \xi_n^{(i)}), i = 1..d, \xi_j^{(i)} \in \overline{\mathbb{K}(z)^G} \).

- Each \( \xi^{(i)} \) has replacement property: \( r(z_1, \ldots, z_n) \in \mathbb{R}(z)^G \Rightarrow r(z_1, \ldots, z_n) = r(\xi_1^{(i)}, \ldots, \xi_n^{(i)}) \).
Example: $SE_2(\mathbb{R}) \sim \mathbb{R}^4$ (second jet bundle of plane curves).

- group:
  
  \[ G = (\lambda_1^2 + \lambda_2^2 - 1) \subset \mathbb{R}[\lambda_1, \lambda_2, \lambda_3, \lambda_4], \quad (\lambda_1 = \cos \phi, \lambda_2 = \sin \phi) \]

  \[ A = \begin{pmatrix}
  Z_1 - \lambda_1 z_1 - \lambda_2 z_2 + \lambda_3, & Z_2 - \lambda_2 z_1 + \lambda_1 z_2 + \lambda_4, \\
  Z_3 - \frac{\lambda_2 + \lambda_1 z_3}{\lambda_1 - \lambda_2 z_2}, & Z_4 - \frac{z_4}{(\lambda_1 - \lambda_2 z_2)^3}.
  \end{pmatrix} \]

- graph: \( O = \langle (1 + z_3^2)^3 Z_4^2 - (1 + Z_3^2)^3 z_4^2 \rangle = (G + A) \cap \mathbb{R}[z, Z] \cdot \]

  \[ O^e = \langle Z_4^2 - \frac{z_4^2}{(1+z_3^2)^3} Z_3^2 - \frac{z_4^2}{(1+z_3^2)^3} \rangle \subset \mathbb{R}(z)[Z]. \]

- cross-section: \( K = (Z_1, Z_2, Z_3) \)
\[ I^e = \left\langle Z_1, Z_2, Z_3, Z_4^2 - \frac{z_4^2}{(1+z_3^2)^3} \right\rangle \]

- ring of rational invariants: \( \mathbb{R}(z)^G = \mathbb{R} \left( \frac{z_4^2}{(1+z_3^2)^3} \right) \)

- 2 replacement invariants: \( \xi^{(\pm)} = (\xi_1^{(\pm)}, \xi_2^{(\pm)}, \xi_3^{(\pm)}, \xi_4^{(\pm)}) = \left( 0, 0, 0, \pm \frac{z_4}{(1+z_3^2)^{3/2}} \right) \)

\[ \frac{z_4}{(1+z_3^2)^{3/2}} = \frac{\xi_4^{(\pm)}}{(1 + \xi_3^{(\pm)})^{3/2}} \]
THANK YOU!
Common algebraic groups:

Notation:
\( \mathbb{K} \) is a field, \( \mathbb{K}^* = \mathbb{K} - \{0\} \) (a group under multiplication)
\( M(n, \mathbb{K}) = \{n \times n \text{ matrices over } \mathbb{K}\} \) (not a group!)

Groups:

- general linear: \( GL(n, \mathbb{K}) = \{A \in M(n, \mathbb{K}) | \det(A) \neq 0\} \)
- special linear: \( SL(n, \mathbb{K}) = \{A \in GL(n, \mathbb{K}) | \det(A) \neq 1\} \)
- orthogonal: \( O(n, \mathbb{K}) = \{A \in GL(n, \mathbb{K}) | AA^T = 1\} \)
- special orthogonal: \( SO(n, \mathbb{K}) = \{A \in O(n, \mathbb{K}) | \det A = 1\} \)
• Euclidean (rigid motions:) $E(n, \mathbb{K}) = O(n, \mathbb{K}) \rtimes \mathbb{K}^n$

• special Euclidean: $SE(n, \mathbb{K}) = SO(n, \mathbb{K}) \rtimes \mathbb{K}^n$

• similarity: $E(n, \mathbb{K}) \times \mathbb{K}^*$

• affine: $A(n, \mathbb{K}) = GL(n, \mathbb{K}) \rtimes \mathbb{K}^n$

• equi-affine (or special affine): $SA(n, \mathbb{K}) = SL(n, \mathbb{K}) \rtimes \mathbb{K}^n$

• projective: $PGL(n, \mathbb{K}) = GL(n, \mathbb{K})/\mathbb{K}^*$
\textbf{$G$-curvature under reparametrization}

Let $\gamma(\tau) = (x(\tau), y(\tau)), \ \tau \in [c, d]$ be reparametrization $\tilde{\gamma}(t) = (x(t), y(t)), \ t \in [a, b]$, i. e.

$\exists \phi : [c, d] \rightarrow [a, b]$ such that $\gamma(\phi(\tau)) = \tilde{\gamma}(\tau)$.

- Curvature is a well defined function from $\text{Im}(\gamma) = \text{Im}(\tilde{\gamma})$ to $\mathbb{R}$:

  $\xi|_{\gamma(\phi(\tau))} = \xi|_{\tilde{\gamma}(\tau)},$

- but the formulae for $\xi|_{\gamma(t)}$ and $\xi|_{\tilde{\gamma}(\tau)}$ are different