Reducing Second-Order Input-Output Equations*

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October 21, 2010

A continuous-time input-output equation is of the form

$$y^{(n)}(t) = h\left(y(t), \ldots, y^{(n-1)}(t), u(t), u^{(1)}(t), \ldots, u^{(n-1)}\right), \tag{1}$$

where h is a meromorphic function, $y^{(i)}(t)$ and $u^{(j)}(t)$ stand for the ith derivative of $y(t)$ and jth derivative of $u(t)$, respectively.

This poster is motivated by studying accessibility of continuous-time and discrete-time input-output equations [1, 2, 3, 4].

To begin with, we consider continuous-time input-output equations of second-order, and assume that h in (1) is a rational function. We are concerned with the following question:

When $n = 2$, can one rewrite (1) *as:*

$$\begin{cases} \phi(z, z^{(1)}) = 0 \\ z = g(y, y^{(1)}, u) \end{cases} \tag{2}$$

where $\phi \in \mathbb{C}[Y_0, Y_1]$ and $g \in \mathbb{C}\left(y, y^{(1)}, u\right)$.

We take a module-theoretic approach to studying this question. We associate a differential field K to (1), and let $\mathcal{S}$ be the ring of linear differential operators over K. Moreover, let (Ω, d) be the K-linear space of $\mathbb{C}$-differentials of K. The K-space Ω is also a left module over $\mathcal{S}$. This module structure connects the derivation of K with $\mathbb{C}$-differentials.

Definition 1. *We say that $g \in K$ is an* autonomous element *of (K, d) if $\mathrm{d}g$ is a torsion element of Ω.*

Observe that g is an autonomous element if and only if (1) can be rewritten as (2). Hence, we study the following two questions:

- deciding if Ω is torsion-free;
- deciding if there exists an autonomous element, and finding one when it is existent.

The first question can be settled by a gcld-computation over K; while the second appears difficult.

*This work was supported in part by a grant of the National Natural Science Foundation of China. (No. 60821002).

Example 2. *Consider a continuous-time equation* $y^{(2)} = u^{(1)}y + uy^{(1)}$. *Its associated field is* $K = \mathbb{C}(y, y^{(1)}, u, u^{(1)} \ldots)$ *with the derivation operator* $\delta(y^{(1)}) = u^{(1)}y + uy^{(1)}$. *Then* $\omega = \mathrm{d}y^{(1)} - u\mathrm{d}y - y\mathrm{d}u$ *is a torsion element of the module of Kähler differentials of* K *over* $\mathbb{C}$. *In this case,* 1 *happens to be an integrating factor of* ω. *Hence, the given equation can be rewritten as* $\delta(y^{(1)} - uy) = 0$.

As the second question is difficult, we describe a differential field extension $\widehat{K}$ of K, and a $\mathbb{C}$-derivation $\hat{\mathrm{d}}$ from $\widehat{K}$ to a module $\widehat{V}$ such that there exists an autonomous element of $(\widehat{K}, \hat{\mathrm{d}})$ if Ω is not torsion-free.

This work grew out of discussions with M. Halás, Ü. Kotta and C. Yuan. We thank them for helpful comments and suggestions.

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