Darboux integrating factors of planar polynomial vector fields: Inverse problems

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Abstract: Given a planar polynomial differential equation

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned}$$

one says that this system admits a Darboux integrating factor if there exist (irreducible, pairwise relatively prime) polynomials $f_1, \ldots, f_r$ and (nonzero) constants $d_1, \ldots, d_n$ such that

$$\operatorname{div} f_1^{-d_1} \cdots f_r^{-d_r} \begin{pmatrix} P \\ Q \end{pmatrix} = 0$$

Integrating factors of this type are interesting for several reasons. For instance, they are (by a result of S. Lie) directly related to certain nonlinear symmetries of the differential equation, and by a now classical result of Prelle and Singer the existence of a Darboux integrating factor with rational exponents is necessary for the existence of an elementary first integral.

One verifies that the existence of a Darboux integrating factor $f_1^{-d_1} \cdots f_r^{-d_r}$ implies for every i the existence of a polynomial L_i such that

$$P \cdot f_{i,x} + Q \cdot f_{i,y} = L_i f_i;$$

in other words the (complex) zero set of f_i is invariant for the differential equation. Thus, any discussion of Darboux integrating factors includes a discussion of algebraic invariant curves.

There have been a number of results on existence and nonexistence of invariant algebraic curves for a given polynomial vector field; most notably a recent computational approach by Coutinho and Schechter.

Inverse problems are also of interest, for computational and for structural reasons: Thus, given $f_1, \ldots, f_r$ and $d_1, \ldots, d_r$ determine all polynomial vector fields with the corresponding prescribed integrating factor; as a preliminary problem, determine all polynomial vector fields with prescribed invariant curves given by $f_1 = 0, \ldots, f_r = 0$. These inverse problems were essentially solved by C. Christopher, J. Llibre, C. Pantazi and the speaker in recent years, in the following sense: (i) In the linear space of all vector fields admitting prescribed invariant curves, there exists a subspace of “trivial” vector fields such that the quotient modulo this subspace has finite dimension. The quotient can be determined, in principle, by standard Groebner base methods. (ii) In the linear space of all vector fields admitting a prescribed Darboux integrating factor, there exists a subspace of “trivial” vector fields such that the quotient modulo this subspace has finite dimension. To determine the quotient computationally, one has to work with standard problems in commutative algebra and, at a critical point, with the first Weyl algebra.

The talk will focus on these inverse problems, and outline some of the methods and open questions.