Arithmetic of
The Algebraic and Differential Generic Galois Group of
A \( q \)-difference Equation

(joint work with C. Hardouin)

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Let \( k \) be a perfect field and \( K \) be a finite extension of \( k(q) \). The Grothendieck conjecture on \( p \)-curvatures asserts that the solutions of a linear differential equation \( L \) with coefficients in \( K(x) \) with \( K \) a number field are algebraic if and only if the \( p \)-curvatures of the equation \( L \) equals zero for almost all prime \( p \) of \( K \). We prove a discrete analog of this conjecture. In the case of \( q \)-difference equations, \( i.e. \), we prove the equivalence among the following facts:
1. a \( q \)-difference module over \( K(x) \) is trivial or equivalently a \( q \)-difference equation \( Y(qx) = A(x)Y(x) \) where \( A \in G_{l}(K(x)) \);
2. its specialization at \( q = \xi \) has zero curvature for almost all primitive roots of unity \( \xi \);
3. its specialization at \( q = \xi \) is endowed with a (necessarily trivial) structure of iterated \( \xi \)-difference module, for almost all primitive roots of unity \( \xi \).

The equivalence between 1. and 3. is an analog of a Matzat-van der Put conjecture for differential equations over field of positive characteristic.

Then we consider two kinds of Galois groups (the second one only under the assumption that \( k \) has zero characteristic) attached to a \( q \)-difference module \( \mathcal{M} \) over \( K(x) \):
- the generic (also called intrinsic) Galois group in the sense of [Kat82] and [DV02], which is an algebraic group over \( K(x) \);
- the generic differential Galois group, which is a differential algebraic group in the sense of Kolchin, associated to the smallest differential tannakian category generated by \( \mathcal{M} \), equipped of the forgetful functor.

The result above leads to an arithmetic description of the generic algebraic (resp. differential) Galois group: it is the smallest algebraic (resp. differential) group containing the curvatures of the \( q \)-difference module for almost all primitive roots of unity \( \xi \). Although no general Galois correspondence holds in this setting, if the characteristic of \( k \) is positive and the generic Galois group is nonreduced, we can prove some devissage.

By specialization of the parameter \( q \) at 1 in the Galois group, we obtain an upper bound for the generic Galois group of the differential equation obtained by specialization. This upper bound has a curvature characterization in the spirit of the Grothendieck-Katz conjecture, but \emph{via} different curvatures than the ones appearing in the conjecture.

References


