Introduction to D-module Theory. Algorithms for Computing Bernstein-Sato Polynomials

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1 Basic notations

Assume $\mathbb{K}$ to be a field of characteristic 0. By R_n we denote the ring of polynomials $\mathbb{K}[x_1, \ldots, x_n]$ in n variables over $\mathbb{K}$ and by D_n we denote the ring of $\mathbb{K}$-linear partial differential operators with coefficients in R_n, that is the n-th Weyl algebra. The ring D_n is the associative $\mathbb{K}$-algebra generated by the partial differential operators ∂_i and the multiplication operators x_i subject to relations

$$\big\{\partial_i x_j = x_j \partial_i + \delta_{ij},\ x_j x_i = x_i x_j,\ \partial_j \partial_i = \partial_i \partial_j \mid 1 \leq i, j \leq n\big\}.$$

That is, the only non-commuting pairs of variables are (x_i, ∂_i); they satisfy the relation $\partial_i x_i = x_i \partial_i + 1$. Finally, denote $D_n[s] = D_n \otimes_{\mathbb{K}} \mathbb{K}[s]$.

2 Main object to study

Let us recall Bernstein's construction. Given $f \in R_n$ a non-zero polynomial, we consider $M = R_n[s, \frac{1}{f}] \cdot f^s$ which is by definition the free $R_n[s, \frac{1}{f}]$-module of rank one generated by the formal symbol f^s. Then M has a natural structure of left $D_n[s]$-module. Here the differential operators act in a natural way,

$$\partial_i\big(g(s,x) \cdot f^s\big) = \left(\frac{\partial g}{\partial x_i} + s g(s,x) \frac{\partial f}{\partial x_i} \frac{1}{f}\right) \cdot f^s \ \in M.$$

Theorem 2.1 (Bernstein)**.** *For every non-constant polynomial $f \in R_n$, there exists a non-zero polynomial $b(s) \in \mathbb{K}[s]$ and a differential operator $P(s) \in D_n[s]$ such that*

$$P(s) f \cdot f^s = b(s) \cdot f^s \ \in \ R_n\Big[s, \frac{1}{f}\Big] \cdot f^s = M. \tag{1}$$

The monic polynomial $b(s)$ of minimal degree satisfying (1) is called the **Bernstein-Sato polynomial** or the **global b-function**.

3 Abstract

Bernstein-Sato polynomial of a hypersurface is an important invariant in Singularity Theory with numerous applications. It is known, that it is complicated to obtain it computationally, as a number of open questions and challenges indicate. In this talk we overview the main properties of this polynomial as well as several well-known algorithms to compute it.

In the second part of the talk, we propose a family of algorithms called `checkRoot` for optimized check of whether a given rational number is a root of Bernstein-Sato polynomial and the computations of its multiplicity. These algorithms are applied in numerous situations.

1. Computation of the b-functions via upper bounds. We use several techniques to find such upper bounds.
   - Embedded resolutions.
   - Topologically equivalent singularities.
   - A'Campo's formula / Spectral numbers.
2. Integral roots of b-functions. They are important in very different settings. We consider these two problems.
   - Logarithmic comparison problem.
   - Intersection homology D-module.
3. Stratification associated with local b-functions. Notably, the algorithm we propose does not employ primary decomposition.
4. Bernstein-Sato polynomials for varieties.

These methods have been implemented in SINGULAR:PLURAL as libraries `dmod.lib` and `bfun.lib`. All the examples that we present have been computed with this implementation.

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