Differential schemes and differential algebraic varieties

We deal with the following three problems.

Differential spectrum of global sections

Let R be an arbitrary differential ring and $X = \operatorname{Spec}^{\Delta} R$ is a differential spectrum of R. One can define a structure sheaf $\mathcal{O}$ on $\operatorname{Spec}^{\Delta} R$ as the inverse image of the structure sheaf on $\operatorname{Spec} R$. The ring of global sections $\Gamma(X)$ is denoted by $\hat{R}$. Differential spectrum of $\hat{R}$ is denoted by $\hat{X}$. The natural morphism $\iota\colon R \to \hat{R}$ induces $\iota^*\colon \hat{X} \to X$ and this mapping is continuous and surjective. The hypothesis is the mapping ι^* is always a homeomorphism.

We prove the following result.

Theorem 1. *Let R be a differential Keigher ring and $D \subseteq \hat{R}$ is a differential subring containing the image of R. Then the contraction mapping*

$$\operatorname{Spec}^{\Delta} D \to \operatorname{Spec}^{\Delta} R$$

is a homeomorphism.

Corollary 2. *Let R be a Ritt algebra. Then the mapping*

$$\iota^*\colon \hat{X} \to X$$

is a homeomorphism.

If we consider a ring R with iterative derivations theorem 1 holds for all differential rings.

Differential integral dependence

There is a notion of integral dependence in commutative algebra. This notion is important because of the following reasons: 1) such homomorphisms appear often if a given morphism is of finite type, 2) such homomorphism has simple geometric description, 3) in terms of integral dependence one can describe universally closed affine schemes.

Unfortunately, in differential case there is no notion of differential integral element. Nevertheless, one can produce a notion of differential integral dependence for differential rings. The differential integral dependence has the same properties as the integral dependence. For example, corresponding morphisms of schemes are closed and preserved under the base extension.

Using this machinery we introduce the following two examples.

Example 3. Let C be an algebraically closed field of characteristic zero with one derivation ∂. Suppose additionally that C consists of constants. Let $C[z]$ be a polynomial ring and $R = C[z]_{(z)}$ is the localization at prime ideal (z). One defines a derivation on R by the rule $\partial(z) = -z^2$. Then R is differentially integral dependent over C. So, this gives us a new effect in differential algebra.

Example 4. Let K be a differentially closed field of characteristic zero with one derivation. Then the differential algebraic variety given by the equation $z' = -z^2$ defines a complete differential algebraic variety X. This result extends the Kolchin result about complete projective differential algebraic varieties.

Catenary property

Let K be a differential field and B is a differentially finitely generated over K algebra. Then one can define a length of a gap between two prime differential ideals. This leads to the notion of "large gap chain". The hypothesis is: If B is an integral domain then for any two prime differential ideals $\mathfrak{q} \subseteq \mathfrak{p}$ the length of any maximal "large gap chain" between $\mathfrak{q}$ and $\mathfrak{p}$ is equal to $\dim^\Delta B/\mathfrak{q} - \dim^\Delta B/\mathfrak{p}$, where $\dim^\Delta B$ is a differential dimension of B over K.

The Rosenfield result says: for every pair of prime differential ideals $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\mathfrak{p}$ is regular in $B/\mathfrak{q}$ with respect to some ranking, there exists a maximal "large gap chain" between them. Jonson introduced the notion of regular prime ideal of B without using the notion of ranking. Every regular prime ideal satisfies the following condition.

Condition 5. Let $A = B_\mathfrak{p}$ be the corresponding local ring with the maximal ideal $\mathfrak{m}$, then the associated graded ring $G_\mathfrak{m}(A)$ is isomorphic to the symmetric algebra $S_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

Every regular with respect to some ranking prime differential ideal is regular in Johnson sense and the condition on an associated graded ring is strictly weaker then the regularity in each sense. Our result is the following.

Theorem 6. *Let $\mathfrak{q} \subseteq \mathfrak{p} \subseteq B$ be a pair of prime differential ideals such that condition 5 holds for the image of $\mathfrak{p}$ in $B/\mathfrak{q}$. Then there exists a maximal "large gap chain" between $\mathfrak{q}$ and $\mathfrak{p}$ of the length* $\dim^\Delta B/\mathfrak{q} - \dim^\Delta B/\mathfrak{p}$.