Talk Abstracts

On Applications of Differential Elimination to Modeling Problems in Biology

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In this talk, I will present applications of differential algebra [10, 8] to applied mathematics problems with a focus towards biological modeling. The methods that I will present rely intensively on differential regular chains computations.

Differential regular chains [9] are generalizations of differential characteristic sets, which were much studied by Wen-Tsün Wu, as well from a theoretical [13] as from an applied point of view [11, 12].

Considered applications include a reduction method for the differentiation index of differential-algebraic systems, a symbolic preprocessing for a parameters estimation problem [6] and a rigorous method for performing quasi-steady state approximations of parametric dynamical systems derived from chemical reaction systems by the principle of the mass-action law. This last application turns out to be very fruitful for modeling gene regulatory networks by means of nonlinear differential equations [4] which is one of the available approaches [5]. Indeed, it applies to the generalized form of the mass-action law [7] and permits to compute reduced models of gene regulatory networks [2, 3].

The talk will be illustrated by demonstrations performed using the new MAPLE *DifferentialAlgebra* package which is a MAPLE interface for the BLAD libraries [1]. The BLAD libraries are open source C libraries dedicated to differential elimination. They aim at widening the use of differential elimination methods by modelers and practitioners.

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A New Symbolic Method for Linear Boundary Value Problems Using Groebner Bases

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(Joint work with Markus Rosenkranz, Georg Regensburger, Loredana Tec)

Boundary Value Problems are of utmost importance for science and engineering. In fact, in nature, problems describable by differential equations hardly occur without boundary conditions. However, surprisingly, even for the linear case, only ad hoc methods for the symbolic solution of such problems are known so far, see the packages provided in Mathematic and Maple.

In the textbooks, the solution of linear boundary value problems is presented by Green's functions that are constructed by certain heuristics for the problems at hand.

In the talk, we present a new method - based on the speaker's Groebner bases theory - for obtaining Green's functions in a uniform way for arbitrary linear boundary value problems.

For the implementation of the method, we use the functor concept introduced by the speaker for the Theorema system. This allows easy adjustment of the code to various domains of coefficients and various representations of the objects in the abstract mathematical domains in which the theory lives.

We conclude by giving an example of a linear boundary value problem that cannot be handled appropriately by the current methods in Mathematica and Maple but can be solved by our new method.

Recent Developments on μ-Basis of Rational Curves and Surfaces

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The μ-basis of rational curves and surfaces is a nice algebraic tool introduced about ten years ago in Geometrical Modeling. The μ-basis provides a link between parametric form and implicit form of rational curves and surfaces, and it has interesting applications in Computer Aided Geometric Design and Geometrical Modeling. In this talk, I will discuss the latest development of mu-basis theory and its applications in surface implicitization, singularity computation of rational curves/surfaces, surface/surface intersections, etc.

Keywords: μ-basis, rational curves and surfaces, implicitzation, singularity, surface/surface intersection.

The Extended Zeilberger's Algorithm with Parameters

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We use the extended Zeilberger's algorithm due to Paule to derive three term recurrences and the structure relations for orthogonal polynomials. Based on this algorithm, we develop a computer algebra approach to proving identities on Bernoulli polynomials and Euler polynomials. In order to employ the extended Zeilberger's algorithm, we use the contour integral definitions of the Bernoulli and Euler numbers so that we can establish recurrence relations on the integrands. Such recurrence relations have certain parameter free properties which lead to the required identities without computing the integrals. We also note that this approach can be applied to prove identities on hypergeometric series and basic hypergeometric series.

JGEX - the System Java Geometry Expert

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JGEX is mainly used to generate proofs of geometry theorems with algebraic methods and human-readable proof methods. The readable proofs are presented in a visually dynamic way so that the user can "see"the proof vividly. We will use various kinds of examples to demonstrate JGEX.

This is joint work with Xiao-Shan Gao, and Zheng Ye.

Homology Rigidity of Exceptional Grassmannians

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Combining the theory of Groebner basis with the Schubert presentation for the cohomology of Grassmannians , we generalize the homology rigidity of the classical Grassmaniann to the exceptional cases.

Involutive Groebner Bases in Boolean Rings

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The fact that there is a close link between involutive systems of differential equations in the Janet-Riquier theory and Groebner bases was established first by Wu Wen-Tsun in 1991. In the present talk we discuss a specialization of our general involutive algorithm which stemmed from the Janet-Riquier theory to a (multivariate) Boolean ring. The output of the algorithm is an involutive basis which is a redundant Groebner basis. In particular, we consider the Pommaret and Janet bases. The former basis can be computed directly in the Boolean ring whereas for the Janet basis one must perform computation in the commutative polynomial ring over F_2. We show that the involutive bases technique is a suitable algorithmic tool for the counting problem in Boolean rings. We also present a new implementation of the Pommaret basis algorithm and compare it with our previous implementation as well as with some other accessible software packages oriented to construction of Boolean Groebner bases.

Geometry of Seki Takakazu (1642? –1708) and Takebe Brothers with the Use of Resultants

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Deciding Feasibility of Polynomial Inequalities by Numerical Filtering

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Deciding the feasibility (existence of solution) of system of polynomial inequalities is a fundamental problem in computational real algebraic geometry. Furthermore, it has many applications in science and engineering. Thus, there have been extensive research on the problem. However, many important and challenging application problems are still practically out of reach for the existing algorithms in spite of tremendous progress made in their efficiency during last 60 years. In this talk, we will describe an efficient numerical "filter"that can decide the feasibility for "most"cases. We need to call the existing symbolic algorithms only when the filter fails (which is rare).

This is joint work with Douglas Marks.

Algebra of Differential Invariants

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Whether algebraic or differential, one can distinguish two families of applications for invariants of group actions: equivalence problems, together with classification and canonical forms, and symmetry reduction. In this latter case invariants are used to take into account the symmetry of a problem, mainly in order to reduce its size or its analysis. The computational requirements include then four main components: the explicit computation of a generating set of invariants, and the relations among them (syzygies); procedures for rewriting the problem in terms of the invariants; and finally procedures for computing in the algebra of invariants. For differential invariants, those issues have been coherently addressed in a series of papers. The algebraic foundations developed therein support an algorithmic suite that is implemented in the Maple package AIDA that works on top of the library DifferentialGeometry and the extension of the library diffalg to non commutative derivations.

In this talk I focus on three descriptions of the differential algebra of differential invariants as given by generators and syzygies. The normalized and edge invariants were the focus in the reinterpretation of the moving frame method by Fels & Olver (1999). My contribution here is first to show the completeness of a set of syzygies for the normalized invariants that can be written down with minimal information on the group action (namely the infinitesimal generators). Second, I provide the adequate general concept of edge invariants and show their generating properties. The syzygies for edge invariants are obtained by applying the algorithms for differential elimination that I generalized to non-commuting derivations. Another contribution is to exhibit the generating and rewriting properties of Maurer-Cartan invariants. Those have desirable properties from the computational point of view. They are all the more meaningful when one understands that they are the differential invariants that come into play in the moving frame method as practiced by Griffiths (1974) and differential geometers. The syzygies for the Maurer-Cartan invariants naturally follow from the structure equations for the group.

Supersparse Interpolation: Mathematics + Algorithmic And Computational Thinking = Mathematics Mechanization

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Algorithms as mathematical objects are as ancient as axioms, proofs and theorems. High-speed electronic computers, created in Professor Wu's lifetime, can apply algorithms to large inputs and thus make possible modern life. Efficient algorithms and computations have impact on mathematics itself, as they open new possibilities but also give rise to new inherent algorithmic inefficiencies.

My talk demonstrates the underlying methodology of symbolic computation on new algorithms, jointly discovered with Michael Nehring, for efficiently interpolating multivariate supersparse rational functions with integer coefficients from modular values. Supersparse (also called lacunary) polynomials and rational functions have a very sparse term structure with degrees hundreds and thousands decimal digits long. The algorithm, as presented, is conjectured to be polylogarithmic in the degrees, but exponential in the number of terms. Therefore, it is very effective for rational functions with tens of non-zero terms, such as the ratio of binomials, but it becomes less effective for a high number of terms. We will demonstrate the computational performance of our algorithm on a laptop computer.

Our algorithms are oblivious to whether the numerator and denominator have a common factor. The algorithm will recover the sparse form of the rational function, rather than the reduced form, which could be dense.

On Wu's Perspective on Theorem Proving with a Recent Application to Program Analysis

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In his seminal papers on automated geometry theorem proving, Wu Wentsun proposed a different perspective on theorem proving. Instead of attempting to prove a theorem in Frege-Hilbert's style, he advocated finding, in addition to a proof of the statement, subsidiary conditions ruling out degeneracies in the statement. The talk will briefly review the impact of Wu's perspective on my subsequent work in automated theorem proving and its applications. Particularly, I will focus on our recent work on strengthening invariants associated with programs, starting with a weak invariant.

Solving Structured Linear Problems in Exact and Approximate Arithmetic

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Problems defined in terms of polynomial equations can often be defined in terms of linear systems of equations where the coefficient matrix has certain special structure. The Euclidean algorithm and its associated linear system with a Sylvester coefficient matrix is the best known example of such a problem. In this talk we show how such problems are often efficiently solved using fraction-free algorithms (in the case of exact arithmetic) and look-ahead methods (in the case of approximate arithmetic). The common thread in both cases is often simply the difference in how one normalizes solutions to recursive structured subproblems.

Theorem Proving in Geometry and Tools for Polynomial System Solving

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There are mainly two classes of computerized mathematical proofs.

The first class consists in automatizing and/or formalizing the proof of already known theorems. Professor Wu Wen-Tsun has initiated this trends in classical geometry and has been followed by many searchers like Shang-Ching Chou, Xiao-Shan Gao and many others. An important aspect of this trend is the formal proof of algorithms and it should be noted that Buchberger algorithm for Groebner bases has been formally proved (including the two Buchberger criteria).

The second class contains proof which are too difficult to be conducted or discovered by hand written reasoning. This is the case of the well known four colors theorem. Many of these proofs are combinatorial nature, like four colors theorem, but the progress of algebraic solving tools allows to conduct such proofs in geometry.

In this talk, we present such a recent proof in geometry, which needed the most recent version of several solving tools.

The trisector of three lines in the three dimensional space is the curve of the points which are at the same Euclidean distance of the three lines. The lines are said in general position if they are not pairwise coplanar nor all parallel to the same plane.

Theorem: The trisector of three lines in general position is either a non singular quartic or the union of a skew cubic and a line which do not intersect. In both cases the trisector is homeomorphic to four skew lines. The trisector contains a line if and only if the hyperboloid which contains them is of revolution.

For the last assertion only, a short hand written proof has been discovered, but only after the computerized proof.

The tools used for the proof were primarily Groebner bases and algorithms for finding at least a point by connected component of an algebraic hypersurface. Several other tools would be useful if there were efficient enough; these are primary decomposition and radical computation. Two other tools allowed (a posteriori) an alternative proof of some steps: The algorithms to decompose a non negative polynomial into a sum of squares and the representation by Wu Wen-Tsun characteristic sets of equidimensional and equiprojetable ideals; however this representation has been computed with Groebner bases in this case.

By presenting the proof of above theorem, it will be shown how such difficult computations allow to compare the algorithms not only by their efficiency but also by the quality of their output and the adequacy of their specifications.

A Method to Solve Algebraic Equations up to Multiplicities via Ritt-Wu's Characteristic Sets

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Ritt-Wu's method of characteristic sets is an effective method to solve algebraic equations by computer, while one cannot get the multiplicity of an isolated solution by this method so far. We restate two definitions for the multiplicity of an isolated solution, they are equivalent and coincide with that of Van der Waerden and Weil. One of the definitions is in the framework of nonstandard analysis, and another is in the framework of standard analysis by using differential topology. To prove the equivalence of them with that of Van der Waeden and Weil, the nonstandard one is essential. By using Ritt-Wu's method on computer again to the algebraic equations with some independent infinitesimals as parameters obtained by modifying the original equations, we can then determine the multiplicity of an isolated solution of the original equations.

An example by computer is given: first get 8 solutions, then compute their multiplicities – 6 of them are simple roots, while each of the other two has multiplicity 6.

Symbolic Computational Geometry with Advanced Invariant Algebras

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The introduction of Descartes' coordinates to geometry is an epoch-making event in the history of human mind in that it changes the subject from qualitative description to quantitative characterization. However, coordinates are a kind of reference system, and have no geometric meaning by themselves. In applications, methods based on coordinates face two major difficulties, one on geometric explicability, the other on middle expression swell. The great mathematician Leibniz once dreamed of a geometric calculus that deals directly with geometric objects instead of sequences of numbers introduced by Descartes' coordinates. The classical invariant theory developed in the 19th century and rejuvenated in the 1970's faces the same two major difficulties in geometric applications. Only with the introduction of advanced invariants and the establishment of advanced invariant theory can the two difficulties be possibly overcome.

This talk introduces the recently established advanced invariant theory and its applications in geometric computing and reasoning. This theory is composed of null Grassmann-Cayley algebra (NGC), null bracket algebra (NBA), null Geometric Algebra (NGA) and Clifford difference ring (CDR), with NGC providing monomial representations of geometric constructions, NBA providing advanced invariant expressions, NGA and CDR serving as the kernel of advanced invariant computing. Controls are made to the number of terms and the degree of polynomials in each step of algebraic manipulations, so that the results remain to be in factored form and be shortest. In automated reasoning of classical geometries, advanced invariant theory has very nice performance. Among the one hundred or so geometric theorems tested, one third need only one term throughout the proving procedure which is usually composed of three or four steps; the overwhelming majority of the theorems can be proved in two terms, and can have their conclusions strengthened by weakening the hypotheses.

Submersive Rational Difference Systems and Their Accessibility

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Many algebraic concepts such as Ore (skew) polynomial rings, modules over these rings, difference ideals and their associated fields play a prominent role in (nonlinear) control theory. Their applications, in general, require the help of computer algebra software. In this talk, we present a result on rational difference systems, which leads to an algebraic setting for using algorithms for Ore polynomials to determine the formal accessibility and to simplify transfer functions of nonlinear discrete-time systems.

The talk reports work joint with Miroslav Halás, Ülle Kotta, Huaifu Wang and Chunming Yuan.

Triangular Decomposition of Polynomial Systems: Algorithmic Advances and Remaining Challenges

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The Characteristic Set Method of Wu has freed Ritt's decomposition from polynomial factorization, opening the door to a variety of discoveries in polynomial and differential algebra. The landmark paper "A Zero Structure Theorem for Polynomial Equations Solving" where the method is proposed already suggests important directions for further developments.

A first part of this talk is an overview of the algorithmic advances which have extended the work of Wu on triangular decomposition. The aim is to highlight the key ideas which have brought either better implementation techniques and practical performance, or a better understanding of the relations between the computed algebraic objects and the represented geometrical entities.

In a second step a collection of questions on the theory and practice of triangular decomposition will be posed. Tentative answers will be sketched for some of them.

On the Mechanization of Geometry:
From Theorem Proving to Knowledge Management

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Mechanizing geometry has been the objective of many great scientists, from ancient times to the information era. Mechanical theorem proving in geometry was one of the most active areas of research in artificial intelligence in the late 1950s, a few years after the advent of modern computers. This area has been well developed in the last three decades under the influence of Wu's work.

Now methods and software packages for geometric theorem proving have reached a good level of maturity, while computing devices have become more and more powerful. This motivates us to study the mechanization of other aspects of geometry than theorem proving. This talk will present some of the aspects centered around the concept and management of geometric knowledge. We view geometric theorems and proofs as well as proof methods as knowledge objects and thus as part of the geometric knowledge. We are interested in creating reliable software environments in which different kinds of geometric knowledge are integrated, effective algorithms and techniques for managing the knowledge are implemented, and the user can use the built-in knowledge data and functions to develop new tools and to explore geometry visually, interactively, and dynamically. To meet such requirements, we have considered and investigated several foundational and engineering issues, including:

1. identification, formalization, structuring, and specification of geometric knowledge objects;
2. standardization, organization, representation, and creation of geometric knowledge data;
3. design and implementation of geometric-object-oriented languages for symbolic geometric computation and reasoning;

4. development of techniques and algorithms for knowledge data processing, automated theorem proving, and geometric diagram generation;

5. display and management of geometric knowledge in the form of dynamic geometric documents.

Some of the key strategies that have been adopted to deal with these issues are:

1. encapsulating knowledge data into knowledge objects and organizing knowledge objects by modeling their hierarchic structure of relations;

2. using predicate logic with embedded knowledge to formalize geometric concepts, theorems, and other knowledge objects;

3. handling geometric degeneracy and uncertainty by means of case distinction.

We will explain and discuss the above-mentioned issues and strategies and demonstrate the effectiveness of the strategies by some pieces of software that have implemented preliminary and experimental versions of our geometric knowledge base, geometric-object-oriented language, and geometric documentation system.

The Mathematics of Calligraphy

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A Dynamical Decision about the Nonnegativity of Multivariate Polynomials

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A dynamical method, successive difference substitution, was proposed few years ago for deciding the nonnegativity of polynomials with nonnegative variables, without cell decomposition. This talk sketches the progress on the topic since then. A new program is demonstrated here with examples. The basic procedure is, split the variables into smaller nonnegative quantities, collect the terms and see whether the coefficients all are nonnegative. This procedure would be repeated a great many times if necessary.

Computation of the Splitting Field of a Polynomial Using its Galois Group

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Computation of the splitting field of a polynomial can be considered as special case of the prime decomposition of an ideal generated by a triangular sets, since the splitting field is expressed as the residue class ring of a multivariate polynomial ring factored by a maximal ideal which is a prime component of the ideal generated by the triangular set, called Cauchy moduli, derived from a given polynomial.

Recent progress on Galois group computation, one can compute the Galois group very efficiently based on modular technique, and by using this computation, one can develop very efficient computation of the Groebner basis of a maximal component of the Cauchy moduli which gives the splitting field.

In this talk, several modular techniques based on the knowledge of the Galois group for efficient computation of the splitting field are explained

with emphasis on usage of multi-approximations of roots such as p-adic approximations of different primes p and numerical approximation, focusing on the correctness of the computation.

祝贺吴文俊先生九十华诞学术会议

数学机械化与教育技术

张景中

(1) 来自数学机械化的思想和成果, 已用于智能教育软件的研究和开发;

(2) 智能教育软件的应用，能够减轻教师的负担，提高学生的学习兴趣；

(3) 数学机械化可能对教育技术的发展产生影响；

(4) 教育技术对数学机械化研究提出了许多新的课题。

关键词：数学机械化，教育技术，智能教育软件。

Mathematics Mechanization and Education Technology

(In honor of Professor Wen-Tsun Wu's ninetieth birthday)

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(1) The thoughts and results from Mathematics Mechanization，have been used in the researching and developing of intelligent software for education.

(2) With the applying of intelligent education software，the burden of teachers would be lighting and the interesting for students in learning would be raising.

(3) The influence of Mathematics Mechanization on the developing of Education Technology would be obviously.

(4) Many new topics have been proposed by Education Technology to Mathematics Mechanization.

Keywords: Mathematics Mechanization, Education Technology, intelligent software for education.

Determining Singular Solutions of Polynomial Systems via Symbolic-numeric Reduction to Geometric Involutive Form

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We present a method based on symbolic-numeric reduction to geometric involutive form to compute the primary component and a basis of Max Noether space of a polynomial system at an isolated singular solution. The singular solution can be known exactly or approximately. If the singular solution is known with limited accuracy, then we propose a generalized quadratic Newton iteration to refine it to high accuracy.

This is joint work with Xiaoli Wu.
