Complete Numerical Isolation of Real Zeros in General Triangular Systems*

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Abstract. We consider the computational problem of isolating all the real zeros of a zero-dimensional triangular polynomial system $F_n \subseteq \mathbb{Z}[x_1, \ldots, x_n]$. We present a complete numerical algorithm for this problem. Our system F_n is general, with no further assumptions. In particular, our algorithm is the first to successfully treat multiple zeros in such systems. A key idea is to introduce evaluation and separation bounds, which are used in conjunction with sleeve bounds to detect zeros of even multiplicity. We have implemented our algorithm and promising experimental results are shown.

Keywords. *system of polynomial equations, triangular polynomial system, zero-dimensional system, isolating interval, real zero isolation, complete numerical algorithms, sleeve bound, evaluation bound.*

1 Introduction

Many problems in the computational sciences and engineering can be reduced to the problem of solving polynomial equations. There are two basic approaches to solving such polynomial

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systems – numerically or algebraically. Usually, the numerical methods have no global guarantees of correctness. Algebraic methods for solving polynomial systems include Gröbner bases [6], characteristic sets [19, 15], CAD (Cylinder Algebraic Decomposition) [2, 3], or resultants [1, 17]. One general idea in polynomial equation solving is to reduce the original system into a triangular system. Zero-dimensional polynomial systems are among the most important cases to solve. This paper considers this case only.

A zero-dimensional triangular system of polynomials has the form $F_n = \{f_1, \ldots, f_n\}$, where each $f_i \in \mathbb{Z}[x_1, \ldots, x_i]$ $(i = 1, \ldots, n)$. We are interested in real zeros of F_n. A real zero of F_n is $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ such that $F_n(\xi) = 0$, i.e.,

$$f_1(\xi_1) = f_2(\xi_1, \xi_2) = \cdots = f_n(\xi_1, \ldots, \xi_n) = 0. \quad (1)$$

The standard idea here is to first solve for $f_1(x_1) = 0$, and for each solution $x_1 = \xi_1$ of f_1, we find the solutions of $x_2 = \xi_2$ of $f_2(\xi_1, x_2) = 0$, etc. This means that the problem can be reduced to solving univariate polynomials of the form

$$f_i(\xi_1, \ldots, \xi_{i-1}, x_i) = 0. \quad (2)$$

Such polynomials have algebraic number coefficients. We could isolate roots of such polynomials by using standard root isolation algorithms, but using algebraic number arithmetic. But even for $n = 2$ or 3, such algorithms are too slow. The numerical approach is to replace the ξ_i's by approximations, and thus reduce the problem to isolating roots of such numerical polynomials. The challenge is how to guarantee completeness of such numerical algorithms.

Results of This Paper. We will provide a numerical algorithm that solves such triangular systems completely in the following precise sense: given an n-dimensional box $R = J_1 \times \cdots \times J_n \subseteq \mathbb{R}^n$ where J_i are intervals, and any precision $\varepsilon > 0$, it will isolate the zeros of F_n in R to precision ε. To isolate the zeros of F_n in R means to compute a set of pairwise disjoint n-dimensional boxes such that each zero of F_n in R is contained in one of these boxes, and each box contains just one zero of F_n. These boxes have diameter bounded by ε.

Our solution places no restriction on F_n. In particular, ours is the first to achieve complete root isolation in the presence of multiple zeros. All the existing algorithms require the system F_n to be square-free (no multiple zeros) and some require F_n to be regular[1] or even irreducible. As is well known, it is expensive to make a triangular polynomial system to be square-free, regular or irreducible.

Many algorithms that seek to provide "exact numerical" solution assume computation over the rational numbers $\mathbb{Q}$. But this is much less efficient than using dyadic numbers: let $\mathbb{D} := \mathbb{Z}[\frac{1}{2}] = \{m2^n : m, n \in \mathbb{Z}\}$ denote the set of dyadic numbers (or bigfloats). Most current fast algorithms for bigfloats can be derived from Brent's work [5]. In the following, we use the symbol $\mathbb{F}$ to denote either $\mathbb{D}$ or with $\mathbb{Q}$. The only computational assumption about $\mathbb{F}$ we need are: (1) the ring operations $(+, -, \times)$ and $x \mapsto x/2$ (halving) are computed without error, and (2) comparisons of the elements of $\mathbb{F}$ are exact. The algorithms of this paper can be implemented exactly over $\mathbb{F}$. We use intervals to isolate real numbers: let $\square\mathbb{F}$ denote the

[1] F_n is **regular** if for each zero $(\xi_1, \ldots, \xi_n)$, the leading coefficient of the polynomial $f_i(\xi_1, \xi_2, \ldots, \xi_{i-1}, x_i)$ does not vanish.

set of intervals of the form $[a, b]$ where $a \le b \in \mathbb{F}$. Note that the assumptions (1) and (2) are different than Brent's axioms [5]; see [21, 22] for an axiomatic treatment for $\mathbb{F}$ in real approximations.

Given a polynomial $f \in \mathbb{R}[X]$ and an interval $I = [a, b] \in \square\mathbb{F}$, the basic idea is to construct two polynomials $f^u, f^d \in \mathbb{F}[X]$ such $f^u > f > f^d$ holds in I. We call (f^u, f^d) a **sleeve** of f over I. We show that if the **sleeve bound** $SB_I(f^u, f^d) := \sup\{f^u(x) - f^d(x) : x \in I\}$ is sufficiently tight, then isolating the roots of f^u and f^d can lead to isolation of the roots of f. Note that the coefficients of $f^u f^d$ are in $\mathbb{F}$, but f have real coefficients which can be arbitrarily approximated.

Univariate root isolation is a well-developed subject in its own right, with many efficient solutions known (see [10, 12, 13, 14] for some recent work). We can use any of these solutions in our algorithm. The only additional property we require in these univariate solvers is that they handle multiple zeros. It is also easy to classify multiple zeros according to their **parity**: the parity of the root is *even* (resp., *odd*) if the root has even (resp., odd) multiplicity. There are simple ways to modify standard algorithms to satisfy our extra requirements.

The critical idea in this paper is the introduction of **evaluation bounds**. For a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and a subset $I \subseteq \mathbb{R}$, let its **evaluation bound** be

$$EB_I(f) := \inf\{|f(x)| : f'(x) = 0, x \in I\}, \tag{3}$$

and its **separation bound** be

$$\Delta_I(f) := \inf\{|x - y| : f(x) = f(y) = 0,\ x, y \in I,\ x \neq y\}. \tag{4}$$

By definition, the infimum over an empty set is ∞. The subscript I may be omitted when $I = \mathbb{R}$. Although separation bounds are well-known tools in the area of root isolation, the use of evaluation bounds appear to be new. It is the ability to compute lower estimates on $EB_I(f)$ and $\Delta(f)$ that allows us to detect zeros of even multiplicities. In particular, if the following **sleeve-evaluation inequality**

$$SB_I(f^u, f^d) < EB_I(f) \tag{5}$$

holds, then we show how the isolating intervals of $f^u f^d$ can be used to define isolating intervals of f. In order to satisfy this inequality, we need to "refine" our sleeves to yield tighter sleeve bounds. Furthermore, we need to generalize sleeves and (3) to the multivariate case of triangular systems.

A major goal in our algorithmic design is the emphasis on "adaptive" techniques. Informally, adaptivity means that the computational complexity is sensitive to the nature of the input instance, and in typical or nice instances, the complexity is low. Thus, we prefer numerical (iterative) tests which are usually adaptive, over more powerful but non-adaptive algebraic techniques. For instance, in our algorithm where we need determine the sign of a derivative at a point, instead of invoking a Sturm sequence computation, we will introduce a simple numerical iteration whose halting condition provided through root separation estimates.

Literature Survey. The idea of using a sleeve to solve equations was used by [18] and [16]. Lu et al [16] proposed an algorithm to isolate the real roots of triangular polynomial system. Their method could solve many problems in practice. But their algorithm is not complete in the sense that it does not have a termination condition and cannot handle multiple zeros. Collins et al [8] considered the problem with interval arithmetic methods and Descartes' method using floating point computation. Based on the CAD method, they considered isolating the real roots of a squarefree triangular system. They constructed a bitstream interval for each real coefficient of a univariate polynomial $f = f_i(\xi_1, \ldots, \xi_{i-1}, X)$. Then they obtain an interval polynomial for f. The sign determination of $f_i(\xi_1, \ldots, \xi_{i-1}, X)$ can be replaced by determining the sign of the two corresponding endpoints of the interval for each coefficient. In this way, they obtained isolating intervals of the triangular system. They pointed out if a real coefficient is zero (but in some implicit representation), the method will fail. Their system is restricted to be regular. Xia and Yang [20], based on the resultant computation, proposed a method to isolate the real roots of a semi-algebraic set. In fact, they ultimately considered the real root isolation of regular and square-free triangular systems. They mentioned that their method is not complete and will fail in some cases. Our root isolation of real polynomials using sleeves is related to Eigenwillig et al [11] who considered root-isolation for real polynomials with bitstream coefficients. Their algorithm requires f to be squarefree; but we require algebraic coefficients when f is non-squarefree. Their algorithm is based on the Descartes method, but ours can be viewed as a generic reduction of the root isolation problem to univariate root isolation in $\mathbb{F}[X]$. Our evaluation bound is analogous the curve separation bounds in Yap [23], who used them to provide the first complete subdivision algorithm for detecting tangential intersection of Bezier curves.

Overview of Paper. In the next section, we describe the basic technique of using sleeves and evaluation bounds of f. We next exploit a special property of sleeves called monotonicity. This leads to an effective criteria for isolating zeros of even multiplicity. Using these tools, we provide an algorithm to isolate the real roots of univariate polynomial with real coefficients. In Section 3, we extend the real root isolation method to multivariate case. We show a method to compute evaluation bound and zero bound, based on a general bound on multivariate zeros. We also show how to construct sleeves and derive a sleeve bound (for $f_i(\xi_1, \ldots, \xi_{i-1}, X)$) that depends on the precision of the given isolating box for $F_{i-1} = \{f_1, \ldots, f_{i-1}\}$. This shows convergence of our algorithm. We also provide greater detail on the refine the isolating boxes. The overall algorithm is also presented here. Section 4 describes some experimental work. We conclude in Section 5.

2 Root Isolation for Real Univariate Polynomials

In this section, we give a framework for isolating the real roots of a univariate polynomial equation with real coefficients.

2.1 Evaluation and Sleeve Bounds

Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{D} := \mathbb{Z}[\frac{1}{2}] = \{m2^n : m, n \in \mathbb{Z}\}$ the set of dyadic numbers, and $\mathbb{F}$ denote either $\mathbb{D}$ or $\mathbb{Q}$. A real function $f : \mathbb{R} \to \mathbb{R}$ is C^1 if it has a continuous derivative $f'(X) = \frac{\partial f}{\partial X}$. In this section, we fix f, f^u, f^d to be C^1 functions, and let $I = [a, b]$ be an interval. In applications later, we will further assume that $f \in \mathbb{R}[X]$, $f^u, f^d \in \mathbb{F}[X]$ and $I \in \square\mathbb{F}$.

We call (I, f^u, f^d) a **sleeve** for f if, for all $x \in I$, we have $f^u(x) > f(x) > f^d(x)$.

For any real function f, let $\mathtt{Zero}_I(f)$ denote the set of distinct real zeros of f in the interval I. If $I = \mathbb{R}$, then we simply write $\mathtt{Zero}(f)$. If $\#(\mathtt{Zero}_I(f)) = 1$, we call I an **isolating interval** of f. Sometimes, we need to count the zeros up to the parity (i.e., evenness or oddness) of their multiplicity. Call a zero $\xi \in \mathtt{Zero}(f)$ an **even zero** if its multiplicity is even, and **odd zero** if its multiplicity is odd. Define the multiset[2] $\mathtt{ZERO}_I(f)$ whose underlying set is $\mathtt{Zero}_I(f)$ and where the multiplicity of $\xi \in \mathtt{ZERO}_I(f)$ is 1 (resp., 2) if ξ is an odd (resp., even) zero of f.

To avoid special treatment near the endpoints of an interval, we would like to enforce the following conditions.

$$|f(a)| \geq EB_I(f), \quad f^u(b)f^d(b) > 0. \tag{6}$$

We say that the sleeve (I, f^u, f^d) is **faithful** for f if (6) as well as the sleeve-evaluation inequality (5) are both satisfied. We can easily see that $|f(a)| \geq EB_I(f)$ implies $f^u(a)f^d(a) > 0$, using (5). We need a stronger condition at $X = a$ than at $X = b$ in (6) because there might be a zero of f just to the left of $X = a$ that can cause confusing in our lemmas below: this asymmetry is a consequence of the monotonicity property below. An appendix will treat the case of non-faithful sleeves.

Intuitively, f is nicely behaved when if we restrict f to a neighborhood of a zero ξ where $|f| < EB(f)$. This is illustrated in Figure 1.

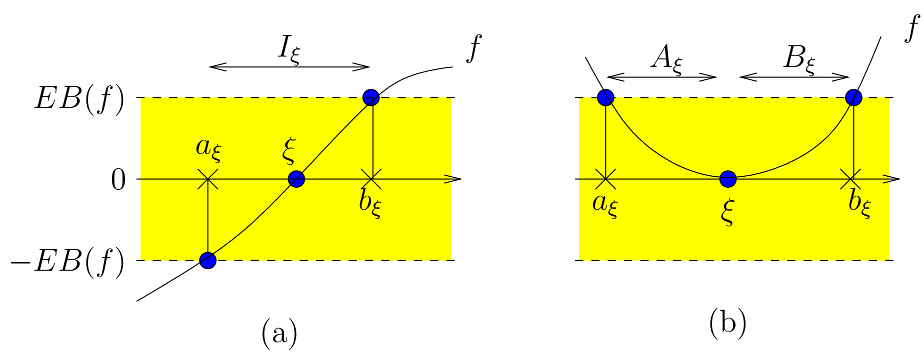

Figure 1: Neighborhood of zero ξ: $I_\xi = A_\xi \cup \{\xi\} \cup B_\xi$.

Given f and I, define the polynomials

$$\widehat{f}(X) := f(X) - EB_I(f), \qquad \overline{f}(X) := f(X) + EB_I(f).$$

[2]A multiset S is a pair (x_S, μ_S) where x_S is a set in the usual sense, and $\mu_S : x_S \to \{1, 2, 3, \ldots\}$ is a function. We call $\mu_S(X)$ the **multiplicity** of $x \in x_S$, and x_S the **underlying set** of S. For simplicity, we write "$x \in S$" instead $x \in x_S$. Also, the **size** of S is defined to be $|S| := \sum_{x \in X} \mu_S(X)$.

If $\xi \in \texttt{Zero}_I(f)$, we define the points a_ξ, b_ξ as follows:

$$\begin{aligned} a_\xi &:= \max\{\max\{a\} \cup \texttt{Zero}(\widehat{f} \cdot \overline{f}) \cap (-\infty, \xi)\}, & (7) \\ b_\xi &:= \min\{\min\{b\} \cup \texttt{Zero}(\widehat{f} \cdot \overline{f}) \cap (\xi, +\infty)\}. & (8) \end{aligned}$$

Then define the open intervals (see Figure 1):

$$A_\xi := (a_\xi, \xi),\ B_\xi := (\xi, b_\xi) \text{ and } I_\xi := (a_\xi, b_\xi). \qquad (9)$$

The basic properties of these intervals are captured here:

LEMMA 1. *Let (I, f^u, f^d) be a faithful sleeve for f. For all $\xi, \zeta \in \texttt{Zero}_I(f)$, we have:*
(i) If $\xi \neq \zeta$ then I_ξ and I_ζ are disjoint.
(ii) $\texttt{Zero}_I(f^u f^d) \subseteq \bigcup_\xi I_\xi$.
(iii-a) $A_\xi \cap \texttt{Zero}(f^u)$ is empty iff $A_\xi \cap \texttt{Zero}(f^d)$ is non-empty.
(iii-b) $B_\xi \cap \texttt{Zero}(f^u)$ is empty iff $B_\xi \cap \texttt{Zero}(f^d)$ is non-empty.
(iv) The derivative f' has a constant sign in A_ξ or B_ξ for any $\xi \in \texttt{Zero}_I(f)$.

Proof. (i) Suppose $\xi < \zeta$ are consecutive zeros of $\texttt{Zero}_I(f)$. Then either f is positive on (ξ, ζ) or f is negative on (ξ, ζ). Wlog, f is positive on (ξ, ζ). Then the multiset $\texttt{ZERO}_I(\widehat{f}) = \texttt{ZERO}(f - EB_I(f))$ has at least two zeros (they may have the same value) in (ξ, ζ). This proves $b_\xi \leq a_\zeta$ and so I_ξ and I_ζ are disjoint.
(ii) Let $z \in \texttt{Zero}_I(f^u f^d)$. Then (5) implies that $|f(z)| < EB_I(f)$. By the definition of evaluation bound, this also means that $f'(z) \neq 0$. Thus there are two cases: either $f(z)f'(z) > 0$ or $f(z)f'(z) < 0$. First, suppose $f(z)f'(z) > 0$. Then there is a unique largest $\xi \in \texttt{Zero}(f)$ that is less than z, and there is a unique smallest $b_\xi \in \texttt{Zero}(\widehat{f})$ that is greater than z. This proves that $z \in (\xi, b_\xi)$. Similarly, if $f(z)f'(z) < 0$, we will see that $z \in (a_\xi, \xi)$ for some $\xi \in \texttt{Zero}_I(f)$.
(iii-a) Either $f(a_\xi) > 0$ or $f(a_\xi) < 0$. If $f(a_\xi) > 0$ then (5) implies $f^d(a_\xi) > 0$. But $f^d(\xi) < 0$, and hence $A_\xi \cap \texttt{Zero}(f^d)$ is non-empty. Now, since f^u is positive over A_ξ, we conclude that $A_\xi \cap \texttt{Zero}(f^u)$ is empty. The other case, $f(a_\xi) < 0$ will similarly imply that $A_\xi \cap \texttt{Zero}(f^d)$ is empty and $A_\xi \cap \texttt{Zero}(f^u)$ is non-empty.
(iii-b) This is similar to (iii-a).
(iv) It is obvious. Otherwise, we assume there exist $s \in A_\xi$ (for B_ξ, the proof is similar) such that $f'(s) = 0$. We derive a contradiction from the definitions of a_ξ by (7), where $A_\xi = (a_\xi, \xi)$. **Q.E.D.**

If $s, t \in \texttt{Zero}_I(f^u f^d)$ such that $s < t$ and $(s, t) \cap \texttt{Zero}_I(f^u f^d)$ is empty, then we call (s, t) a **sleeve interval** of (I, f^u, f^d). The following is immediate from the preceding lemma(iii):

COROLLARY 2. *Each zero of $\texttt{Zero}_I(f)$ is isolated by some sleeve interval of (I, f^u, f^d).*

LEMMA 3. *Let (I, f^u, f^d) be a faithful sleeve. For all $\xi \in \texttt{Zero}_I(f)$, the multiset $\texttt{ZERO}_{B_\xi}(f^u \cdot f^d)$ has odd size. Similarly, the multiset $\texttt{ZERO}_{A_\xi}(f^u \cdot f^d)$ has odd size.*

Proof. We just prove the result for the multiset $\texttt{ZERO}_{B_\xi}(f^u \cdot f^d)$. Wlog, let $f(b_\xi) > 0$ (the case $f(b_\xi) < 0$ is similar). By the sleeve-evaluation inequality, $f^d(b_\xi) > 0$. Note that when

$b_\xi = b$, the inequality is also true since (I, f^u, f^d) is faithful. But $f^d(\xi) < 0$. Hence f^d has an odd number of zeros (counting multiplicities) in the interval $B_\xi = (\xi, b_\xi)$. Moreover, $f^u > f$ implies f^u has no zeros in B_ξ. **Q.E.D.**

It follows from the preceding lemma that for each zero ξ of f, the multiset $\mathtt{ZERO}_{I_\xi}(f^u f^d)$ has even size. Hence the multiset $\mathtt{ZERO}_I(f^u f^d)$ has even size, say $2m$. So we may denote the sorted list of zeros of $\mathtt{ZERO}_I(f^u f^d)$ by

$$(t_0, t_1, \ldots, t_{2m-1}). \tag{10}$$

where $t_0 \leq t_1 \leq \cdots \leq t_{2m-1}$. Note that $t_i = t_{i+1}$ iff t_i is an even zero of $f^u f^d$. Intervals of the form $J_i := [t_{2i}, t_{2i+1}]$ where $t_{2i} < t_{2i+1}$ are called **candidate interval** of the sleeve. We immediately obtain:

Corollary 4. *Each $\xi \in \mathtt{Zero}_I(f)$ is contained in some candidate interval of a faithful sleeve (I, f^u, f^d).*

Proof. We use the notations in (9) and (10), and use ξ to represent a root of f in I. From Lemma 1 (ii), any element of $\mathtt{ZERO}_I(f^u f^d)$ is in some I_ξ. From Lemma 3, $I_\xi \cap \mathtt{ZERO}_I(f^u f^d)$ has even size. Therefore, the smallest element of $A_\xi \cap \mathtt{ZERO}_I(f^u f^d)$ is of the form t_{2k}. From Lemma 3, $A_\xi \cap \mathtt{ZERO}_I(f^u f^d)$ has odd size. Then the largest element of A_ξ is also of the form t_{2s} and the smallest element of B_ξ is t_{2s+1}. As a consequence, ξ is in the candidate interval (t_{2s}, t_{2s+1}). **Q.E.D.**

Which of these candidate intervals actually contain zeros of f? To do this, we classify a candidate interval $[t_{2j}, t_{2j+1}]$ in (10) into two types:

$$\left.\begin{array}{ll} \text{(Odd):} & t_{2j} \in \mathtt{Zero}(f^d) \text{ if and only if } t_{2j+1} \in \mathtt{Zero}(f^u) \\ \text{(Even):} & t_{2j} \in \mathtt{Zero}(f^d) \text{ if and only if } t_{2j+1} \in \mathtt{Zero}(f^d) \end{array}\right\} \tag{11}$$

Thus we call a candidate interval J an **odd** or **even candidate interval** depending on whether it satifies (11)(Odd) or (11)(Even). We now treat the easy case of deciding which candidate intervals are isolating intervals of f:

Lemma 5 (Odd Zero). *Let J be a candidate interval. The following are equivalent:*
(i) J is an odd candidate interval.
(ii) J contains a unique zero ξ of f. Moreover ξ is an odd zero of f.

Proof. Let $J = [t, t']$.
(i) implies (ii): Wlog, let $f^u(t) = 0$ and $f^d(t') = 0$. Thus, $f(t) < 0$ and $f(t') > 0$. Thus f has an odd zero in J. By Corollary 2, we know that candidate intervals contain at most one distinct zero.
(ii) implies (i): Since ξ is an odd zero, we see that f must be monotone over J. Wlog, assume f is increasing. This implies $f^d(t) < 0$ and hence $f^u(t) = 0$. Similarly, $f^u(t') > 0$ and hence $f^d(t') = 0$. Hence J is an odd candidate. **Q.E.D.**

Lemma 5 provides the theoretical basis to isolate zeros of odd multiplicity. Isolate zeros of even multiplicity is more subtle and will be dealt with in the following section. To do this we need to look at the sign of derivatives of f^u and f^d. We make a first observation along this line:

LEMMA 6. *Let $t_i \in$ ZERO$(f^u f^d)$.*
(a) If t_i is a zero of f^u,
then i is even implies $\frac{\partial f^u}{\partial X}(t_i) \geq 0$, and i is odd implies $\frac{\partial f^u}{\partial X}(t_i) \leq 0$.
(b) If t_i is a zero of f^d,
then i is even implies $\frac{\partial f^d}{\partial X}(t_i) \leq 0$. and i is odd implies $\frac{\partial f^d}{\partial X}(t_i) \geq 0$.

Proof. The result is true for $i = 0$, using faithfulness. The rest follows by induction based on parity tracking. **Q.E.D.**

2.2 Monotonicity Property

We will now exploit a special property of sleeve (I, f^u, f^d) for f:

$$\frac{\partial f^u}{\partial X} \geq \frac{\partial f}{\partial X} \geq \frac{\partial f^d}{\partial X} \qquad \text{holds in } I \tag{12}$$

We call this the **monotonicity property**. In this subsection, we assume the monotonicity property (12) and as well the faithfulness of the sleeve.

We now strengthen one half of Lemma 3 above.

LEMMA 7. *For all $\xi \in \texttt{Zero}_I(f)$, there is a unique zero of odd multiplicity of $f^u \cdot f^d$ in $A_\xi = (a_\xi, \xi)$.*

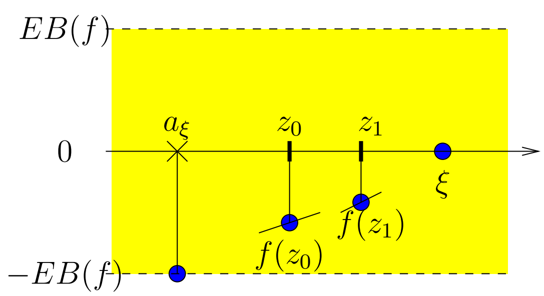

Figure 2: A_ξ has a unique zero of $f^u \cdot f^d$: CASE of $f^u(z_0) = f^u(z_1) = 0$.

Proof. Alternatively, this lemma says that the multiset $\texttt{ZERO}_{A_\xi}(f^u f^d)$ has size 1.

By way of contradiction, suppose $z_0 \leq z_1$ are two zeros of $f^u f^d$ in $A_\xi = (a_\xi, \xi)$. Note that we allow the possibility that $z_0 = z_1$ (in which case z_0 is an even root of $f^u f^d$). From Lemma 1(iii), we know that either $z_0, z_1 \in \texttt{ZERO}(f^u)$ or $z_0, z_1 \in \texttt{ZERO}(f^d)$ (i.e., it is not possible that one is a zero of $\texttt{ZERO}(f^u)$ and the other is a zero of $\texttt{ZERO}(f^d)$). There are two cases:

(A) z_0, z_1 are roots of f^u. See Figure 2. By Rolle's theorem, there exists $z \in [z_0, z_1]$ such that $\frac{\partial f^u}{\partial X}(z) = 0$. Therefore, there exist $z^- < z < z^+$ that are arbitrarily close to z such that

$$\frac{\partial f^u}{\partial X}(z^-) \cdot \frac{\partial f^u}{\partial X}(z^+) < 0. \tag{13}$$

On the other hand, note that $f(z_j) < f^u(z_j) = 0$ for $j = 0, 1$. Since $f(\xi) = 0$, and $z_j < \xi$, this means that the interval (z_j, ξ) contains a point z with $f'(z) > 0$. But f' has constant sign in A_ξ from Lemma 1 (iv), and so this sign of f' is positive. Then by monotonicity (12),

$$\frac{\partial f^u}{\partial X}(z^-) \geq f'(z^-) > 0, \quad \text{and} \quad \frac{\partial f^u}{\partial X}(z^+) \geq f'(z^+) > 0. \tag{14}$$

Now we see that (13) and (14) are contradictory.

(B) z_0, z_1 are roots of f^d. We similarly derive a contradiction. **Q.E.D.**

Remark: It should be observed that this lemma does not hold when A_ξ is replaced by B_ξ. This somewhat surprising asymmetry can be seen in the proof of the preceding result.

Corollary 8. *If t_{2j} is an even zero of $f^u f^d$, then $J_j = [t_{2j}, t_{2j+1}]$ contains no zero of f.*

Proof. If J_j contains a zero ξ of f, then t_{2j} would be an even zero of $f^u f^d$ contained in A_ξ, contradicting Lemma 7. **Q.E.D.**

If t_{2j} is an even zero we have either $t_{2j} = t_{2j+1}$ or $t_{2j} = t_{2j-1}$. But the former case only give us a trivial candidate interval which clearly has no zeros of f. The next result is a useful consequence of monotonity:

Lemma 9. *The interval $J_0 = [t_0, t_1]$ is a candidate interval and it isolates a zero of f.*

Proof. Faithfulness implies $|f(a)| \geq EB_I(f)$. By symmetry, consider the case $f(a) \leq -EB_I(f)$. Then $t_0 \in \mathtt{Zero}(f^u)$. Moreover, $t_0 \in A_\xi$ where ξ is the first zero of f. By the previous lemma, t_0 is an odd zero and $t_0 < t_1$. If J_0 is an odd candidate, the lemma is true. So assume J_0 is an even candidate and wlog let $t_0, t_1 \in \mathtt{Zero}(f^d)$. By Lemma 6, $\frac{\partial f^d}{\partial X}(t_1) \geq 0$ and so by monotonicity, $\frac{\partial f}{\partial X}(t_1) \geq 0$. On the other hand, $f(t_0) > f^d(t_0) = 0$ and faithfulness implies $f(a) > EB(f) > f(t_0)$ and so $\frac{\partial f}{\partial X} < 0$ in A_ξ. This implies J_0 contains ξ as an even zero, as desired. **Q.E.D.**

In Lemma 5, we showed that (11)(Odd) holds iff J_j isolates an odd zero of f. The next result shows what condition must be added to (11)(Even) in order to to characterize the isolation of even zeros.

Lemma 10 (Even Zero). *Let $J_j = [t_{2j}, t_{2j+1}]$ be an even candidate interval.*
Then J_j isolates an even zero ξ of f iff one of the following conditions hold:
(i) $f^d(t_{2j}) = 0$ and $\frac{\partial f^u}{\partial X}(t_{2j}) < 0$
(ii) $f^u(t_{2j}) = 0$ and $\frac{\partial f^d}{\partial X}(t_{2j}) > 0$.

Note: if $j > 0$ in this lemma, then t_{2j-1} is a zero of f^d iff t_{2j} is a zero of f^d.

Proof. Let t_{2j} be a zero of f^d (if it is a zero of f^u, the proof is similar). So $f^d(t_{2j+1}) = 0$ and by Lemma 6, $\frac{\partial f^d}{\partial X}(t_{2j+1}) \geq 0$. Then monotonicity implies $\frac{\partial f}{\partial X}(t_{2j+1}) \geq 0$. Next, $t_{2j+1} \in B_\xi$ for some zero ξ of f. This means $\frac{\partial f}{\partial X}$ is positive in the interval (ξ, t_{2j+1}). There are two cases: (a) $t_{2j} < \xi < t_{2j+1}$ or (b) $\xi < t_{2j} < t_{2j+1}$. If (a), then since $f(t_{2j}) > f^d(t_{2j}) = 0$, we conclude that $\frac{\partial f}{\partial X}(t_{2j}) < 0$ (see Figure 3(a)). If (b), then $\frac{\partial f}{\partial X}(t_{2j}) > 0$ since $\frac{\partial f}{\partial X}$ has constant sign in B_ξ (see Figure 3(b)). **Q.E.D.**

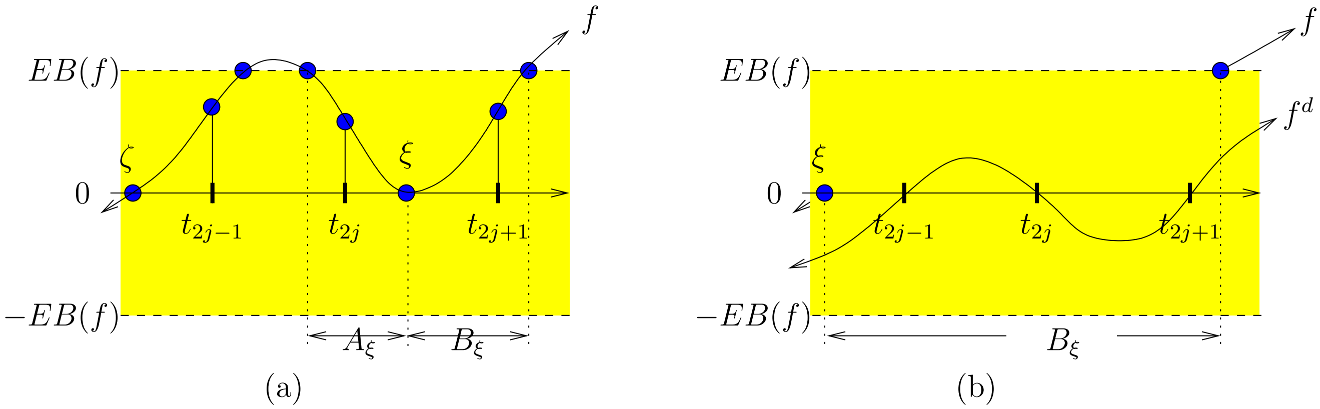

Figure 3: Detection of even zero when $t_{2j}, t_{2j+1} \in \mathtt{Zero}_I(f^d)$: (a) even zero, (b) no zero

2.3 Effective Root Isolation of f

So far, we have been treating the roots t_j of $f^u f^d$ exactly. But in our algorithms, we only have isolating intervals $[a_i, b_i]$ of these t_j's. We now want to replace the candidate intervals $[t_{2i}, t_{2i+1}]$ by their "effective versions" of the form $[a_{2i}, b_{2i+1}]$. As usual, we assume that our sleeve (I, f^u, f^d) is faithful and satisfies the monotonicity property (12). Let $\mathtt{ZERO}_I(f^u f^d)$ be the sorted list given in (10), and $[a_i, b_i]$ an isolating interval of t_i, where any two distinct intervals $[a_i, b_i]$ and $[a_j, b_j]$ are disjoint. Let

$$SL_{f,I} = ([a_0, b_0], [a_1, b_1], \ldots, [a_{2m-1}, b_{2m-1}]) \tag{15}$$

be the corresponding list of isolating intervals for the roots of $f^u f^d$ in $\mathtt{ZERO}_I(f^u f^d)$. Assume that $[a_i, b_i] = [a_j, b_j]$ iff $t_i = t_j$. Note that $t_i = t_j$ implies $|i - j| \leq 1$. Let

$$K_i := [a_{2i}, b_{2i+1}]. \tag{16}$$

By Corollary 8, J_i is not an isolating interval if t_{2i} is an even zero. Hence, we call K_i an **effective candidate** iff $t_{2i} < t_{2i+1}$ and t_{2i} is an odd zero. Thus, K_i contains the candidate interval $J_i = [t_{2i}, t_{2i+1}]$. Furthermore, K_i is called an **effective even candidate** (resp., **effective odd candidate**) if J_i is an even (resp., odd) candidate interval (cf. (11)).

Our next theorem characterizes when K_i is an isolating interval of f. This is the "effective version" of Lemma 5 and Lemma 9. But before this theorem, we provide a useful partial criterion in the case when K_i is an effective even candidate:

LEMMA 11. *Let $K_i = [a_{2i}, b_{2i+1}]$ be an effective even candidate. Then K_i isolates an even zero provided one of the following conditions hold:*
(E')^d: $t_{2i} \in \mathtt{Zero}_I(f^d)$ and $\frac{\partial f}{\partial X}$ is negative at a_{2i} or b_{2i+1}.
(E')^u: $t_{2i} \in \mathtt{Zero}_I(f^u)$ and $\frac{\partial f}{\partial X}$ is positive at a_{2i} or b_{2i+1}.

Proof. Say t_{2i} is a zero of f^d (the case where $t_{2i} \in \mathtt{Zero}(f^u)$ is similarly shown). By Lemma 9, there is a zero ξ of f in $[t_{2i}, t_{2i+1}]$ iff $\frac{\partial f}{\partial X}(t_{2i}) < 0$ iff there exists $x \in [t_{2i}, t_{2i+1}]$ such that $\frac{\partial f}{\partial X}(x) < 0$.

...

We have either $t_{2i} \in A_\xi$ (A) or $t_{2i} \in B_\xi$ (B), for some $\xi \in \texttt{Zero}(f)$. Now (A) implies $\frac{\partial f}{\partial X}(x) < 0$ for some $x \in K_i$ (A'), and (B) implies $\frac{\partial f}{\partial X}(x) > 0$ for all $x \in K_i$ (B'). Since (A') is the negation of (B'), this means (A) iff (A'), and similarly (B) iff (B'). **Q.E.D.**

THEOREM 12. *Let $K_i = [a_{2i}, b_{2i+1}]$ be an effective candidate. If K_i is an even effective candidate, further assume that $b_{2i} - a_{2i} < \Delta(f')$. Then K_i is an isolating interval of f iff one of the following cases hold:*
(O) K_i is an effective odd candidate.
(E)^d: K_i is an effective even candidate, and $f^d(t_{2i} = 0$ and $\frac{\partial f}{\partial X}$ is negative at a_{2i} or b_{2i}.
(E)^u: K_i is an effective even candidate, and $f^u(t_{2i} = 0$ and $\frac{\partial f}{\partial X}$ is positive at a_{2i} or b_{2i}.

Proof. As a preliminary remark, we note that K_i contains at most one zero of f. To see this, since $K_i = [a_{2i}, t_{2i}] \cup [t_{2i} \cup t_{2i+1}] \cup [t_{2i+1}, b_{2i+1}]$, and $[t_{2i}, t_{2i+1}]$ is a candidate interval, it suffices to show that $[a_{2i}, t_{2i}]$ and $[t_{2i+1}, b_{2i+1}]$ has no zero of f. If K_i is the first (or the last) effective candidate interval, it is clear that there is no root of f in $[a_{2i}, t_{2i}]$ (or $[t_{2i+1}, b_{2i+1}]$). Else, we have $t_{2i-1} < t_{2i}$ (since t_{2i} is an odd zero), and so f has no zeros in $[t_{2i-1}, t_{2i}] \supseteq [a_{2i}, t_{2i}]$ since these are non-candidate intervals. Similarly, if $t_{2i+1} < t_{2i+2}$ then f has no zeros in $[t_{2i+1}, t_{2i+2}] \supseteq [t_{2i+1}, b_{2i+1}]$. It is possible that $t_{2i+1} = t_{2i+2}$, but again f has no zeros in the non-candidate interval $[t_{2i+2}, t_{2i+3}] \supseteq [t_{2i+1}, b_{2i+1}]$. This completes our justification that K_i has at most one zero.

Suppose K_i is an effective odd candidate. Then Lemma 5 shows that K_i is isolating. Suppose K_i is an effective even candidate. Assume $f^d(t_{2i}) = 0$ (the case $f^u(t_{2i}) = 0$ is similar). Then the previous lemma shows if f' is negative at a_{2i} or b_{2i} then K_i is isolating. Conversely, suppose K_i is isolating. We claim that f' is negative at a_{2i} or b_{2i}. Suppose otherwise: $f'(a_{2i}) \geq 0$ and $f'(b_{2i}) \geq 0$. By Lemma 9, that $f'(t_{2i}) < 0$. This implies that $f'(x) = f'(y) = 0$ for some $x \in [a_{2i}, t_{2i})$ and $y \in (t_{2i}, b_{2i}]$. This is a contradiction since $|x - y| \leq b_{2i} - a_{2i} \leq \Delta(f')$. **Q.E.D.**

3 Bounds for Triangular Systems

In this section, we generalize the univariate evaluation and sleeves for a univariate polynomial to a triangular polynomial system F_n where

$$F_n = \{f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, \ldots, x_n)\} \tag{17}$$

where $f_i \in \mathbb{Z}[x_1, \ldots, x_i]$. Generalizing our univariate notation, if $B \subseteq \mathbb{R}^n$, let $\texttt{Zero}_B(F_n)$ denote the set of real zeros of F_n restricted to B.

Let $B = I_1 \times \cdots \times I_n$ be a n-dimensional box, $I_i = [a_i, b_i]$, and $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \square\xi = I_1 \times \cdots \times I_{n-1}$ be a real zero of $F_{n-1} = \{f_1, \ldots, f_{n-1}\} = 0$. Consider the polynomial

$$f(X) := f_n(\xi_1, \ldots, \xi_{n-1}, X). \tag{18}$$

We have a three-fold goal in this section:

- Compute lower estimates on the evaluation bound $EB_{I_n}(f)$ and separation bound $\Delta_{I_n}(f)$.
- Compute a sleeve (I_n, f^u, f^d) for f that satisfies the monotonicity property.
- Compute an upper estimate on the sleeve bound $SB_{I_n}(f^u, f^d)$.

3.1 Lower Estimate on Evaluation and Separation Bounds

We will give two methods to estimate lower bounds on the evaluation bound $EB_{I_n}(f)$. One method is based on a general result about multivariate zero bounds in [22]; another is based on resultant computation.

Let $\Sigma = \{p_1, \ldots, p_n\} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ be a system of n polynomials in n variables. Assume Σ is zero-dimensional, i.e., it has finitely many complex zeros. Let $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ be one of these zeros. Suppose $d_i = \deg(p_i)$ and $K := \max\{\sqrt{n+1}, \|p_1\|_2, \ldots, \|p_n\|_2\}$ where $\|p\|_2$ is the 2-norm of p. Then we have the following result [22, p. 341]:

PROPOSITION 13. *Let $(\xi_1, \ldots, \xi_n)$ be a complex zero of Σ. For any $i = 1, \ldots, n$, if $|\xi_i| \neq 0$ then*

$$|\xi_i| > MRB(\Sigma) := (2^{3/2} N K)^{-D}\, 2^{-(n+1)d_1 \cdots d_n}. \tag{19}$$

where

$$N := \binom{1 + \sum_{i=1}^n d_i}{n}, \quad D := (1 + \sum_{i=1}^n \frac{1}{d_i}) \prod_{i=1}^n d_i.$$

Note that this proposition defines a numerical value $MRB(\Sigma)$ (the **multivariate root bound**) for any zero-dimensional system $\Sigma \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ of n polynomials. We will now exploit such a value for suitable Σ associated with F_n. Given F_n as in (17), consider the set

$$\widehat{F}_n := \{f_1, \ldots, f_{n-1}, \frac{\partial f_n(x_1, \ldots, x_{n-1}, X)}{\partial X}, Y - f_n(x_1, \ldots, x_{n-1}, X)\} \tag{20}$$

of $n+1$ polynomials in $\mathbb{Z}[x_1, \ldots, x_{n-1}, X, Y]$.

LEMMA 14. *Use the notations in (18). Let $(\xi_1, \ldots, \xi_{n-1})$ be a zero of F_{n-1}. Then the evaluation bound of $f(X) := f_n(\xi_1, \ldots, \xi_{n-1}, X) \in \mathbb{R}[X]$ satisfies $EB_{I_n}(f) > MRB(\widehat{F}_n)$.*

Note that our evaluation bound $EB_{I_n}(f)$ in this lemma is a global one: it does not depend on the interval I_n. We do not know any general method to exploit this I_n.

It is instructive to directly define the **evaluation bound** of a triangular system F_n: for $B \subseteq \mathbb{R}^n$, let $B' = B \times \mathbb{R}$. Then define

$$EB_B(F_n) := \min\{|y| : (x_1, \ldots, x_{n-1}, x, y) \in \mathtt{Zero}_{B'}(\widehat{F}_n), y \neq 0\}. \tag{21}$$

If the set that which we are minimizing over is empty, then $EB_B(F_n) = \infty$. Observe that (21) is a generalization of the corresponding univariate evaluation bound (3). Note that for $i = 2, \ldots, n$, we similarly have evaluation bounds $EB_{B_i}(F_i)$ for F_i, where $F_i = \{f_1, \ldots, f_i\}$.

This multivariate evaluation bound is a lower bound on the univariate one: with f given by (18), we have

$$EB_{I_n}(f) \geq EB_B(F_n).$$

Furthermore, we have

$$EB_B(F_n) > MRB(\widehat{F}_n).$$

Since $MRB(\widehat{F}_n)$ is easily computed, our algorithm can use $MRB(\widehat{F}_n)$ as the lower bound on $EB(F_n)$.

In general, $MRB(\widehat{F}_n)$ is very small and does not give a good lower bound on the evaluation bound (see the Worked Examples in Section 5 below). We propose a computational way to compute such a lower bound, via resultants. Consider $\widehat{F}_n$ defined by (20). Let

$$e_i = \begin{cases} \mathtt{res}_X(Y - f_n, \frac{\partial f_n}{\partial X}) & i = n, \\ \mathtt{res}_{x_i}(e_{i+1}, f_i) & i = n-1, \ldots, 1 \end{cases} \tag{22}$$

where $\mathtt{res}_x(p, q)$ is the resultant of p and q relative to x. Thus $e_1 \in \mathbb{F}[Y]$. If $e_1 \not\equiv 0$, define

$$R(F_n) := \min\{|z| : e_1(z) = 0, z \neq 0\}.$$

If e_1 has no real roots, let $R(F_n) = \infty$.

LEMMA 15. *If $e_1 \not\equiv 0$, $EB(F_n) \geq R(F_n)$, and we can use $R(F_n)$ as the evaluation bound.*

Proof. From properties of the resultant, $e_1(Y)$ is a linear combination of the polynomials in $\widehat{F}_n$. If $(x_1, \ldots, x_{n-1}, x, y) \in \mathtt{Zero}(\widehat{F}_n)$, y must be a zero of $e_1(Y) = 0$. From (22), we conclude $EB(F_n) \geq R(F_n)$. **Q.E.D.**

Therefore, we may isolate the real roots of $e_1(Y) = 0$ and take $\min\{l_1, -r_2\}$ as the evaluation bound for F_n, where (l_1, r_1) and (l_2, r_2) are the isolating intervals for the smallest positive root and the largest negative root of $e_1(Y) = 0$ respectively.

Lower Estimate on Separation Bound. We similarly need a lower estimate on the separation bound $\Delta(f')$. Consider the system D_n comprising the following polynomials:

$$\begin{array}{l} f_1(x_1), f_2(x_1, x_2), \ldots, f_{n-1}(x_1, \ldots, x_{n-1}) \\ \dfrac{\partial f_n(x_1, \ldots, x_{n-1}, X)}{\partial X} \\ \dfrac{\partial f_n(x_1, \ldots, x_{n-1}, Y)}{\partial X} \\ Z - X + Y \end{array} \tag{23}$$

Thus for any zero $(\xi_1, \ldots, \xi_{n-1}, x', y', z') \in \mathtt{Zero}(D_n)$, we have x', y' are zeros of $f' = \frac{\partial f}{\partial X}$ and f is given by (18). Moreover, $z' = x' - y'$ and so $z' \neq 0$ implies $|z'| \leq \Delta(f')$. This proves that $MRB(D_n)$ is a lower bound on $\Delta(f')$. Again, we could develop a computational analogue of the separation bound.

3.2 Construction of a Sleeve

Our construction depend on I_n only in a very minimal way: we need only to assume a definite sign in I_n. This means $0 \notin I_n$, or equivalently, either $I_n > 0$ or $I_n < 0$. In fact, the construction depends on the signs of each of the intervals $I_1, \ldots, I_{n-1}$. We will assume that $I_i > 0$ for $i = 1, \ldots, n$; below we indicate how to reduce the general case to this "positive" case.

Given a polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$, we may decompose it uniquely as

$$g = g^+ - g^-$$

where $g^+, g^- \in \mathbb{R}[x_1, \ldots, x_n]$ each has only positive coefficients, and the support of g^+ and g^- are both minimum. Here, the support of a polynomial g is the set of power products with non-zero coefficients in g.

Given f as in (18) and an isolating box $\Box\xi \in \Box\mathbb{F}^{n-1}$ for ξ, following [16, 18], we define

$$\begin{array}{rcl} f^u(X) & := & f_n^u(\Box\xi, X) = f_n^+(b_1, \ldots, b_{n-1}, X) - f_n^-(a_1, \ldots, a_{n-1}, X), \\ f^d(X) & := & f_n^d(\Box\xi, X) = f_n^+(a_1, \ldots, a_{n-1}, X) - f_n^-(b_1, \ldots, b_{n-1}, X) \end{array} \tag{24}$$

where $f_n = f_n^+ - f_n^-$, $\Box\xi = I_1 \times \cdots \times I_{n-1}$, and $I_i = [a_i, b_i]$ for each i.

We briefly indicate two possible solutions when our assumption that $I_i > 0$ fails. Perhaps the simplest is to shift the origin of F_n so that the box $I_1 \times \cdots \times I_n$ lies in the first quadrant of $\mathbb{R}^n$. E.g., replace x_i by $x_i - a_i$ in F_n and replace I_i by $a_i + I_i$. Alternatively, proceed as follows: for each i, if $\xi_i = 0$, we can replace x_i in $f_n(x_1, \ldots, x_n)$ by 0. After this, we can split I_i if necessary so that $I_i > 0$ or $I_i < 0$. For each i such that $I_i < 0$, we replace x_i in $f_n(x_1, \ldots, x_n)$ by $-x_i$. Let $\bar{f}_n(x_1, \ldots, x_n)$ denote the polynomial after these replacements. Now we may carry out the construction of (23) $\bar{f}_n$ with the box $B' = I'_1 \times \cdots \times I'_n$ where I'_i is $-I_i$ iff $I_i < 0$ and otherwise $I'_i = I_i$.

From the construction, it is clear that

$$f^u \geq f \geq f^d.$$

Moreover, both inequalities are strict if $a_i = \xi_i = b_i$ does not hold for any $i = 1, \ldots, n-1$. Hence $(I_n, f^u(X), f^d(X)$ is a sleeve for $f(X)$ [16, 18]. We further have:

Lemma 16. *Over any positive interval $I_n = [l, r] > 0$, we have:*
(i) (Monotonicity)

$$\frac{\partial f^u}{\partial X} \geq \frac{\partial f}{\partial X} \geq \frac{\partial f^d}{\partial X}.$$

(ii) $f^u(X) - f^d(X)$ is monotonously increasing over I_n.

Proof. Let $f(X) = f_n(\xi_1, \ldots, \xi_{n-1}, X) = f_n^+(\xi_1, \ldots, \xi_{n-1}, X) - f_n^-(\xi_1, \ldots, \xi_{n-1}, X)$ and

$$\begin{array}{rcl} T_1(X) & = & f^u(X) - f(X) \\ & = & (f_n^+(b_1, \ldots, b_{n-1}, X) - f_n^+(\xi_1, \ldots, \xi_{n-1}, X)) \\ & & +(f_n^-(\xi_1, \ldots, \xi_{n-1}, X) - f_n^-(a_1, \ldots, a_{n-1}, X)), \\ T_2(X) & = & f(X) - f^d(X) \\ & = & (f_n^+(\xi_1, \ldots, \xi_{n-1}, X) - f_n^+(a_1, \ldots, a_{n-1}, X)) \\ & & +(f_n^-(b_1, \ldots, b_{n-1}, X) - f_n^-(\xi_1, \ldots, \xi_{n-1}, X)), \\ T_3(X) & = & f^u(X) - f^d(X) \\ & = & (f_n^+(b_1, \ldots, b_{n-1}, X) - f_n^+(a_1, \ldots, a_{n-1}, X)) \\ & & +(f_n^-(b_1, \ldots, b_{n-1}, X) - f_n^-(a_1, \ldots, a_{n-1}, X)). \end{array}$$

Since f_n^+, f_n^- are polynomials with positive coefficients and $0 < a_i \le \xi_i \le b_i$ for all i, $f_n^+(b_1, \ldots, b_{n-1},\ X) - f_n^+(\xi_1, \ldots, \xi_{n-1}, X), f_n^-(\xi_1, \ldots, \xi_{n-1}, X) - f_n^-(a_1, \ldots, a_{n-1}, X)$, and hence $T_1(X)$ are polynomials in X with positive coefficients. Similarly, $T_2(X)$ and $T_3(X)$ are polynomials with positive coefficients. Then when $x > 0$, we have

$$\frac{\partial T_1(x)}{\partial X} = \frac{\partial f^u(x)}{\partial X} - \frac{\partial f(x)}{\partial X} \ge 0,$$

$$\frac{\partial T_2(x)}{\partial X} = \frac{\partial f(x)}{\partial X} - \frac{\partial f^d(x)}{\partial X} \ge 0,$$

$$\frac{\partial T_3(x)}{\partial X} = \frac{\partial f^u(x)}{\partial X} - \frac{\partial f^d(x)}{\partial X} \ge 0.$$

We can directly deduce $\frac{\partial f^u}{\partial X} \ge \frac{\partial f}{\partial X} \ge \frac{\partial f^d}{\partial X}$. As a consequence, $f^u(X) - f^d(X)$ is monotonously increasing in I_n. **Q.E.D.**

As an immediate corollary, we obtain an upper estimate on the sleeve bound:

Corollary 17.

$$SB_{I_n}(f^u, f^d) \le f^u(r) - f^d(r). \tag{25}$$

3.3 Upper Estimate on Sleeve Bound

How good is the upper estimate (24)? Our next goal is to give an upper bound on $f^u(r) - f^d(r)$ as a function of

$$b := \max\{b_1, \ldots, b_n\}, \qquad w := \max\{w_1, \ldots, w_n\}$$

where $w_i = b_i - a_i$. Also let $\mathbf{w} = (w_1, \ldots, w_n)$. For $f \in \mathbb{R}[x_1, \ldots, x_n]$, write $f = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}(x_1, \ldots, x_n)$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and $p_{\boldsymbol{\alpha}}(x_1, \ldots, x_n)$ denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $\|f\|_1 := \max_{\boldsymbol{\alpha}} |c_{\boldsymbol{\alpha}}|$ denote its 1-norm. The inner product of two vectors, say $\mathbf{w}$ and $\boldsymbol{\alpha}$, is denoted $\langle \mathbf{w}, \boldsymbol{\alpha} \rangle = \sum_{i=1}^n w_i \alpha_i$.

Lemma 18. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *where* $m = \sum_{i=1}^n \alpha_i \ge 1$. *Then*

$$p_{\boldsymbol{\alpha}}(b_1, \ldots, b_n) - p_{\boldsymbol{\alpha}}(a_1, \ldots, a_n) \le b^{m-1} \langle \boldsymbol{\alpha}, \mathbf{w} \rangle \le wmb^{m-1}.$$

For example, if each $\alpha_i = m/n$ then $\sum_{i=1}^{n} w_i\alpha_i \le mw/n$.

Corollary 19. *Let* $f = \sum_{\boldsymbol\alpha} c_{\boldsymbol\alpha} p_{\boldsymbol\alpha}(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$.
If each coefficient $c_{\boldsymbol\alpha}$ *is positive and* $m = \deg(f) \ge 1$, *then*

$$\begin{aligned} f(b_1, \ldots, b_n) - f(a_1, \ldots, a_n) &\le b^{m-1} \sum_{\boldsymbol\alpha} |c_{\boldsymbol\alpha}| \langle \mathbf{w}, \boldsymbol\alpha \rangle \\ &\le wmb^{m-1}\|f\|_1. \end{aligned}$$

Theorem 20. *Let* (I_n, f^u, f^d) *be a sleeve for* $f(X)$ *as in ((23)), and* $\Box_{n-1}\xi = I_1 \times \cdots \times I_{n-1}$ *an isolating box for* $\xi \in \mathbb{R}^{n-1}$, *where* $I_i = [a_i, b_i] > 0$, $I_n = [l, r] > 0$, *and* $w = \max_{i=1}^{n-1}\{b_i - a_i\}$. *Then*

$$SB_I(f^u, f^d) \le wm\|f_n\|_1 b^{m-1},$$

where $m = \deg(f_n)$, $b = \max\{b_1, \ldots, b_{n-1}, r\}$.

We give two corollaries to the above theorem.

Corollary 21. *For a fixed* F_n *and* I_n, *when* $w \to 0$, $SB_{I_n}(f^u, f^d) \to 0$.

So when $w \to 0$, $f^u \to f$ and $f^d \to f$. The correctness of our algorithm follows from the fact with sufficient refinement, the sleeve-evaluation inequality (5) will eventually hold. The next corollary gives an explicit condition to guarantee this:

Corollary 22. *The sleeve-evaluation inequality (5) holds provided*

$$w < \frac{EB_{I_n}(f)}{m\|f_n\|_1 b^{m-1}}. \tag{26}$$

4 The Main Algorithm

In this section, we present our root isolation algorithm for a triangular system: given F_n as in (17), to isolate the real zeros of F_n in a given n-dimensional box $B = I_1 \times \cdots \times I_n$. But first, we outline the method for the case $n = 2$. Most of the issues in the general algorithm already appear in this case, but the notations are more transparent. We also give two subalgorithms for root refinement and an effective method for verifying zeros.

4.1 Bivariate Algorithm

This is omitted in the abstract.

4.2 Refinement of Isolating Box

Refining an isolation box is a basic step in our isolation algorithm. Let $\Box_n\xi = \Box_{n-1}\xi \times [c, d] > 0$ be an isolating box for a solution $\xi = (\xi_1, \ldots, \xi_{n-1}, \xi_n)$ of F_n=0, $([c, d], f^d(X), f^u(X))$ a sleeve associated with $\Box_n\xi$ satisfying the sleeve-evaluation inequality (5) and the monotonicity property (12), $\Box'_{n-1}\xi$ an isolating box of F_{n-1} such that $\Box'_{n-1}\xi \subsetneq \Box_{n-1}\xi$, $f(X) = f_n(\xi_1, \ldots, \xi_{n-1}, X)$, and

$$\begin{aligned} \bar{f}^u(X) &= f_n^u(\Box'_{n-1}\xi, X) \text{(for definition, see (23))}, \\ \bar{f}^d(X) &= f_n^d(\Box'_{n-1}\xi, X). \end{aligned}$$

LEMMA 23. *Let t_0, t_1 be the real roots of $f^u f^d = 0$ in $[c, d]$ and $t'_0 < t'_1$ the two smallest real roots of $\bar{f}^u \bar{f}^d = 0$ in $[c, d]$. If $\square'_{n-1} \neq [\xi_1, \xi_1] \times \cdots \times [\xi_{n-1}, \xi_{n-1}]$, then $[t'_0, t'_1] \subset [t_0, t_1]$ and $\xi \in \square'_{n-1}\xi \times [t'_0, t'_1]$.*

Proof. From $\square'_{n-1}\xi \subsetneq \square_{n-1}\xi$, $\square'_{n-1} \neq [\xi_1, \xi_1] \times \cdots \times [\xi_{n-1}, \xi_{n-1}]$, and (23), we have

$$f^d(x) < \bar{f}^d(x) < f(x) < \bar{f}^u(x) < f^u(x), \forall x \in [c, d].$$

It is not difficult to check that sleeve-evaluation inequality (5) and the monotonicity property (12) hold for the sleeve $([c, d], \bar{f}^u, \bar{f}^d)$. Wlog, we assume $f^u(t_0) = 0, f^d(t_1) = 0$. The proofs for other cases are similar. We have $\bar{f}^u(t_0) < f^u(t_0) = 0$ and $\bar{f}^u(\xi_n) > f(\xi_n) = 0$. Then $\bar{f}^u$ has at least one root in (t_0, ξ_n). Since $(t_0, \xi_n) \subset A_{\xi_n}$, by Lemma 7, $\bar{f}^u(x)$ has a unique real root in (t_0, ξ_n). Let t'_0 be this root. Then, $t'_0 > t_0$. Since $\bar{f}^u(x) < f^u(x) < 0$, $\bar{f}^u$ has no real roots in $[c, t_0]$ and t'_0 is the smallest root of $\bar{f}^u \bar{f}^d = 0$ in $[c, d]$. Similarly, we could show that $\bar{f}^d(x) = 0$ has at least one root in (ξ_n, t_1). Let t'_1 be the smallest of these roots. Then t'_0 and t'_1 are the two smallest roots of $\bar{f}^u \bar{f}^d = 0$ in $[c, d]$ and $\xi_n \in (t'_0, t'_1) \subset [t_0, t_1]$. **Q.E.D.**

The lemma tells us how to refine the isolating box of a triangular system without checking which of the subdivided intervals is the isolating interval with Theorem 10.

Refine(F_n, K, ϵ)
Input: $K = I_1 \times \cdots \times I_n$ (an isolating box of the triangular system F_n) and ϵ (a given precision).
Output: A refined isolating box $\hat{K} = \hat{I}_1 \times \cdots \times \hat{I}_n$ of K such that $w = \max_{j=1}^n\{|\hat{I}_j|\} \leq \epsilon$.
1. If $n = 1$, subdivide I_n by half until $|I_n| < \epsilon$ and return I_n.
2. Let $K_{n-1} = I_1 \times \cdots \times I_{n-1}$.
   $w = \max_{j=1}^n\{|I_j|\}$.
   If $w \leq \epsilon$, return K.
   $\delta = \epsilon$.
3. while $w > \epsilon$, do
   3.1. $\delta = \delta/2$.
   3.2. $K_{n-1} :=$ Refine$(F_{n-1}, K_{n-1}, \delta)$.
   3.3. If K_{n-1} is a point, $f(X) := f_n(\xi_1, \ldots, \xi_{n-1}, X)$ is a univariate polynomial with rational coefficients. Subdivide I_n by half until $|I_n| < \epsilon$ and return I_n.
   3.4. Compute the sleeve: $f^u(X) := f_n^u(K_{n-1}, X), f^d(X) := f_n^d(K_{n-1}, X)$.
   3.5. Isolate the real roots of $f^u f^d$ in I_n with precision δ.
   3.6. Denote the first two intervals as $[c_1, d_1], [c_2, d_2]$.
   3.7. $w := d_2 - c_1$.
4. Return $\hat{K} := K_{n-1} \times [c_1, d_2]$.

Remark: In step 3.3, when K_{n-1} is a point, $K_{n-1} = [\xi_1, \xi_1] \times \cdots \times [\xi_{n-1}, \xi_{n-1}] \in \mathbb{F}^{n-1}$, where $(\xi_1, \ldots, \xi_{n-1})$ is the real root of F_{n-1} in $I_1 \times \cdots \times I_{n-1}$.

Proof of Correctness. By Lemma 22, we need only select the first two isolating intervals of $f^d f^u = 0$. By Corollary 20, when $|K_{n-1}| \to 0$, $f^u \to f$ and $f^d \to f$. Since we isolate the real roots of $f^u f^d$ in I_n with precision δ, after enough subdivision, w will be smaller than ϵ and the algorithm will terminate. $\square$

4.3 Verifying Zeros

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be any real root of the zero-dimensional triangular system $\Sigma = \{h_1(x_1), h_2(x_1, x_2), \ldots, h_k(x_1, \ldots, x_k)\}$, $B = I_1 \times \cdots \times I_k$ an isolating box of $\alpha = (\alpha_1, \ldots, \alpha_k)$, and $g(x_1, \ldots, x_k) \in \mathbb{Z}[x_1, \ldots, x_k]$. In this section, we show how to check whether $g(\alpha_1, \ldots, \alpha_k) = 0$.

Consider the following polynomial system:

$$\Sigma_B = \{h_1(x_1), h_2(x_1, x_2), \ldots, h_k(x_1, \ldots, x_k), Y - g(x_1, \ldots, x_k)\}. \tag{27}$$

By Proposition 12, if $g(\alpha_1, \ldots, \alpha_k) \neq 0$, $|g(\alpha_1, \ldots, \alpha_k)| > \rho = MRB(\Sigma_B)$. We call ρ the *zero bound* of $g(\alpha_1, \ldots, \alpha_k)$.

Similar to the computation of evaluation bounds, we have two methods to compute the zero bound. First, by Proposition 12, $\rho = MRB(\Sigma_B)$ can be taken as the zero bound. Second, we may compute the zero bound by resultant computation. Let $r_{k+1} = Y - g(x_1, \ldots, x_k)$ and $r_i = \mathtt{res}(h_i(x_1, \ldots, x_i), r_{i+1}(x_1, \ldots, x_i, Y), x_i)$ for $i = k, \ldots, 1$. Then $r_1(Y)$ is a univariate polynomial in Y. If $r_1 \not\equiv 0$, let $(a_j, b_j), a_j b_j > 0, j = 1, \ldots, s$ be isolating intervals for the nonzero real roots of $r_1(Y) = 0$. We may take $\rho = \min\{|a_j|, |b_j|\}$. The reason is that if $Y_0 = g(\alpha_1, \ldots, \alpha_k) \neq 0$, it must be a nonzero root of $r_1(Y) = 0$ and ρ is smaller than the absolute value of any nonzero root of $r_1(Y) = 0$. Please note that the second method is complete for regular triangular systems only.

We give the following algorithm.

ZeroTest($F_n, K = I_1 \times \cdots \times I_n, I_i = [a_i, b_i], g(x_1, \ldots, x_n)$)
Input: K(an isolating box of a root ξ of triangular system F_n), $g \in \mathbb{F}[x_1, \ldots, x_n]$, $I_i > 0$).
Output: Whether $g(\xi) = 0$.
1. $\delta = \max_{j=1}^{n}\{|I_j|\}$.
2. Compute a sleeve for g:
   $g^u = g^+(b_1, \ldots, b_n) - g^-(a_1, \ldots, a_n)$,
   $g^d = g^+(a_1, \ldots, a_n) - g^-(b_1, \ldots, b_n)$.
3. If $g^d = g^u$, then $g = g^d = g^u$. If $g^d = 0$ return TRUE; otherwise return FALSE. end
4. If $g^u g^d \geq 0$, then $g \neq 0$ and return FALSE. end
5. Compute the zero bound ρ if it does not exist.
6. If $|g^u| < \rho$, and $|g^d| < \rho$, then $g < \rho$ and hence $g = 0$ and return TRUE. end
7. $\delta = \delta/2$, $K = \text{Refine}(F_n, K, \delta)$, and goto step 2.

Proof of Correctness. From the construction, we have $g^d \leq g \leq g^u$. If $g^d = g^u$, then $g = g^d = g^u$ and $g = 0$ iff $g^d = 0$. If $g^u g^d \geq 0$, then $g \neq 0$. Note that $g^d < g < g^u$ in this case. The sign of g is the same as the sign of $\mathtt{sign}(g^u)$ or $\mathtt{sign}(g^d)$. In the two cases, we need not to compute zero bound of g. If $g^u g^d < 0$, we need to compute the zero bound ρ. If $|g^u| < \rho$ and $|g^d| < \rho$, then $g < \rho$ and hence $g = 0$ by the definition of the zero bound. It is obvious that the algorithm will terminate since g^u and g^d approach g when $|K| \to 0$. □

4.4 Isolation Algorithm

We now give the real root isolation algorithm for a triangular system.

RootIsol

Input: $F_n \subseteq \mathbb{Z}[x_1, \ldots, x_n]$, $B_n = \prod_{i=1}^{n} I_i \subset \Box\mathbb{F}_+^n$ with $I_i = [l_i, r_i]$, $\epsilon > 0$.

Output: An isolating set $\Box\mathtt{Zero}_{B_n}(F_n)$.

1. Compute an isolating interval set $\Box\mathtt{Zero}_{B_1}(F_1)$ for F_1 to precision ϵ.
   $Result := \Box\mathtt{Zero}_{B_1}(F_1)$. $NewResult := \emptyset$.
   If $Result = \emptyset$, return $Result$, end
2. For i from 2 to n, do
   2.1. Compute an evaluation bound $EB_i := EB(F_i)$ for F_i.
   2.2. $\delta := \epsilon$.
   2.3. while $Result \neq \emptyset$, do
      2.3.1. Choose an element $\Box_{i-1}\xi$ from $Result$.
         $Result := Result \setminus \{\Box_{i-1}\xi\}$.
      2.3.2. Let $f(X) := f_i(\xi_1, \ldots, \xi_{i-1}, X) = \sum_k c_k(\xi_1, \ldots, \xi_{i-1})X^k$.
         If ZeroTest($F_{i-1}, \Box_{i-1}\xi, c_k(\xi_1, \ldots, \xi_{i-1})$) =TRUE for all k then $f_i \equiv 0$, then the input system is not zero dimensional. end
      2.3.3. Compute the sleeve: $f^u(X) := f_i^u(\Box_{i-1}\xi, X)$, $f^d(X) := f_i^d(\Box_{i-1}\xi, X)$.
      2.3.4. While $f^u(b_i) - f^d(b_i) \geq EB_i$,
         $\delta := \delta/2$.
         $\Box_{i-1}\xi :=$ Refine($F_{i-1}, \Box_{i-1}\xi, \delta$).
         $f^u(X) := f_i^u(\Box_{i-1}\xi, X)$, $f^d(X) := f_i^d(\Box_{i-1}\xi, X)$.
      2.3.5 Isolate the real roots of $f^u f^d$ in I_i.
      2.3.6 Refine the isolating intervals of $f^u f^d$ until the constant sign condition (**??**) holds.
      2.3.7 Compute the parity of each root of $f^u f^d$ in I_i.
      2.3.8 Construct the effective candidate intervals.
      2.3.9 for each effective candidate interval K,
         Apply Theorem 10 to decide whether K is isolating.
         If K is isolating, then
            $K :=$ Refine(F_i, K, ϵ).
            $NewResult := NewResult \bigcup \{\Box_{i-1}\xi \times K\}$.
   2.4. If $NewResult = \emptyset$, return $NewResult$, end
   2.5. $Result := NewResult$. $NewResult := \emptyset$.
3. return $Result$.

Remarks:

1. In Step 2.1, when the system is not regular, we need to compute the evaluation bound by Proposition 12. When the system is not zero-dimensional, we still use Proposition 12 to compute the evaluation bound. Though the evaluation bound is not right, the system can be detected to be nonzero-dimensional in the end. The algorithm is correct in global sense.
2. Algorithm RootIsol can be improved or made more complete in the following ways.

- In the Section 3.3, we give two methods to compute the sleeve bound. Note that the algorithm based on (24) is more adaptive. If we use (26) to give the sleeve bound, the Sleeve-Evaluation Inequality holds automatically.
- Theorem 10 gives criteria to isolate roots in an open interval. We thus need to check whether a rational number r is a solution of $f(X) = 0$. In other words, we need to

check whether $f(r) = f_i(\xi_1, \ldots, \xi_{i-1}, r) = 0$, which can be done with the ZeroTest algorithm.

- We will show that the assumption $B_n > 0$ is reasonable. If we want to obtain the real roots of f in the interval $I = (a, b) < 0$, we may consider $g(X) = f(-X)$ in the interval $(-b, -a)$. If $0 \in (a, b)$, we can consider the two parts, $(a, 0)$ and $(0, b)$ respectively, since we can check if 0 is a root of $f(X) = 0$.

- If we want to find all the real roots of f, we first isolate the real roots of f in $(0, 1)$, then isolate the real roots of $g(X) = X^n * f(1/X)$ in $(0, 1)$, and check whether 1 is a root of f. As a result, we can find all the roots of $f(X) = 0$ in $(0, +\infty)$. We can find the roots of $f(X) = 0$ in $(-\infty, 0)$ by isolating the roots of $f(-X) = 0$ in $(0, +\infty)$. Finally, check whether 0 is a root of $f(X) = 0$.

- Theorem 10 assumes that the sleeves are faithful (see (6)). We will show how to isolate the real roots of f when the sleeve-evaluation inequality (5) holds but $|f(a)| < EB_I(f)$ or $f^u(b)f^d(b) \leq 0$.

  In fact, if we replace $EB_I(f)$ with

$$ET_I(f) := \min\{|f(z)| : z \in \texttt{Zero}_I(f') \cup \{a, b\} \setminus \texttt{Zero}_I(f)\}, \tag{28}$$

  then almost all the sleeve (I, f^u, f^d) is faithful except for $f(a) = 0$ or $f(b) = 0$. If $f(a) = 0$ or $f(b) = 0$, we can ignore the first or last element in $SL_{f,I}$ to form effective candidate intervals of f. When $f(a) = 0$, the first effective candidate interval may or may not be the isolating interval of f, we need to check it by Theorem 10. And we need to use the first isolating interval in $SL_{f,I}$ to decide whether the first effective candidate interval is isolating if the first three elements in $SL_{f,I}$ are all isolating intervals of f^u (or f^d).

  Although we can simply solve the non-faithful problem as mentioned above, when $f(a)$ or $f(b)$ is very close but not equal to 0, $ET_I(f)$ is very small. It is expensive to construct (I, f^u, f^d) in order to satisfy the sleeve-evaluation inequality (5). In order to avoid this case, we just use $EB_I(f)$ directly and deal with the non-faithful sleeve case as in the Appendix.

4.5 Worked Examples

We provide some worked examples with multiple zeros. Note that all the rational numbers in our examples are dyadics ($\mathbb{D}$).

Example 1: Consider the system $F_2 = \{f_1, f_2\}$ where

$$\begin{aligned} f_1 &= x^4 - 3\,x^2 - x^3 + 2\,x + 2, \\ f_2 &= y^4 + xy^3 + 3\,y^2 - 6\,x^2y^2 + 4\,x\,y + 2\,xy^2 - 4\,x^2y + 4\,x + 2. \end{aligned}$$

We omit the details in this abstract.

Example 2: Consider the system $F_3 = \{f_1, f_2, f_3\}$ where

$$\begin{aligned} f_1 &= x^3 - 2\,x^2 + 8, \\ f_2 &= 4\,y^4 + (4\,x^3 - 8\,x^2 - 32)\,y^2 + x^6 - 4\,x^5 + 4\,x^4 + 16\,x^3 - 32\,x^2 + 64, \\ f_3 &= (2\,z^2 + 2\,y^2 + x^3 - 2\,x^2 - 8)^2 + 32\,x^3 - 64\,x^2. \end{aligned}$$

Here $f_3 = 0$ is a surface in $\mathbb{R}^3$ discussed in [4] and [7].

Again, we omit the details in this abstract.

4.6 Experimental Results

In order to evaluate the effectiveness of our algorithms, we implemented RootIsol in Maple 10 and did extensive tests on randomly generated triangular systems. In our implementation, we derive the evaluation bound with the resultant computation method introduced in section 3.1. The most time-consuming parts are the computation of the evaluation bounds for the system and the refinement for the isolating boxes.

We tested our program with three sets of examples. The coefficients of the tested polynomials are within -100 to 100. The precision is set to be $\frac{1}{2^{10}}$. We use the method mentioned in the **Remark** for RootIsol to compute *all the real solutions* for the triangular systems. The timings are collected on a PC with a 3.2G CPU and 512M memory.

The first set of examples are sparse polynomials and the results are given in Table 1. We use the Maple command randpoly($\{x_1, \ldots, x_n\}$, degree=d, terms =t) to generate polynomials with given degree and given number of terms. The *type* of a triangular system $F_n = \{f_1, \ldots, f_n\}$ is a list $(d_1, \ldots, d_n)$ where d_i is the degree of f_i in x_i. The column started with TYPE is the type of the tested triangular systems. TIME is the average running time for each triangular system in seconds. NS is the average number of real solutions for each triangular system. NT is the number of tested triangular systems. NE is the average number of terms in each polynomial.

TYPE	TIME	NS	NT	NE
(3, 3)	0.04862	2.04	100	(4, 10)
(9, 7)	0.52717	3.99	100	(10, 10)
(21, 21)	108.9115	5.45	20	(10, 10)
(35, 31)	1450.9592	9.3	10	(10, 10)
(3, 3, 3)	0.15783	3.48	100	(4, 10, 10)
(9, 7, 5)	16.20573	8.36	100	(10, 10, 10)
(3, 3, 3, 3)	1.69115	5.64	100	(4, 10, 10, 10)
(3, 3, 3, 3, 3)	159.1199	8.0	10	(4, 10, 10, 10, 10)

Table 1: Timings for solving sparse triangular systems

The second set of examples are dense polynomials and the results are given in Table 2. A triangular system $F_n = \{f_1, \ldots, f_n\}$ of type $(d_1, \ldots, d_n)$ is called *dense* if $f_i = \sum_{k=0}^{d_i} c_k x_i^k$ and $\deg(c_k, x_j) = d_j - 1$ for all k and $i > j$.

The third set of test examples are triangular systems with multiple roots and the results are given in Table 3. The triangular system is defined as follows. $f_1 = randpoly(x, degree = d_1, dense)$, $f_i = randpoly(\{x_1, \ldots, x_{i-1}\}, [\frac{d_i}{2}])^2 * (randpoly(\{x_1, \ldots, x_{i-1}\}, degree = d_{i-1} - 1) * x_i + randpoly(\{x_1, \ldots, x_{i-1}\}, degree = d_{i-1} - 1))^{[\frac{d_i+1}{2}] - [\frac{d_i}{2}]}$, $i > 1$, where $[a]$ is the maximal

TYPE	TIME	NS	NT	NE
(3, 3)	0.05355	1.91	100	(3.99, 8.02)
(9, 8)	1.87486	4.26	100	(9.94, 43.98)
(11, 11)	8.78255	4.5	80	(11.975, 72.5)
(16, 14)	50.22294	6.0	100	(16.9, 127.13)
(21, 15)	164.23443	6.22	100	(21.91, 176.8)
(28, 23)	1968.4497	6.0	10	(28.5, 345.4)
(35, 31)	14963.3546	11.0	5	(35.8, 580.0)
(3, 3, 3)	0.38702	2.91	100	(3.99, 7.77, 13.01)
(5, 4, 4)	2.97011	4.88	100	(5.99, 14.72, 24.24)
(5, 5, 5)	33.225275	5.6125	80	(5.9625, 17.775, 42.1375)
(8, 7, 6)	592.1848	7.6	10	(8.9, 36.0, 79.8)
(11, 9, 5)	2987.5606	9.0	5	(12.0, 63.8, 91.0)
(3, 3, 3, 3)	119.94042	6.96	50	(4.0, 8.12, 12.82, 20.92)
(5, 5, 3, 3)	551.4401	3.4	10	(6.0, 32.1, 42.3, 21.5)

Table 2: Timings for solving dense triangular systems

integer which is less than a. In Table 3, NM is the average number of multiple roots for the tested triangular systems.

TYPE	TIME	NS	NM	NT	NE
(5, 5)	0.71251	3.71	1.57	100	(5.97, 34.47)
(9, 8)	0.60408	3.1	3.1	100	(9.94, 18.92)
(13, 11)	32.44376	6.55	3.92	100	(13.94, 107.68)
(23, 21)	466.0289	6.15	3.75	20	(24.0, 183.4)
(3, 3, 3)	3.21342	5.59	3.24	100	(3.99, 13.08, 31.71)
(9, 7, 5)	425.95055	12.95	8.15	20	(9.95, 60.85, 100.35)
(3, 3, 3, 3)	130.617	11.15	6.1	20	(4.0, 12.2, 33.7, 62.95)

Table 3: Timings for solving dense triangular systems

From the above experimental results, we could conclude that our algorithm is capable of handling quite large triangular systems.

5 Conclusion

This paper provides a complete algorithm of isolating the real roots for arbitrary zero-dimensional triangular polynomial systems. The key idea is to use a sleeve satisfying the the sleeve-evaluation inequality to isolate the roots for a univariate polynomial with algebraic number in its coefficients. To achieve this goal, we also developed methods to estimate the evaluation bounds and the sleeve bounds. Even with our current simple implementation, the algorithm is shown to be quite effective. As we mentioned before, to solve larger problems, the bottle neck of the algorithm is the computation of the evaluation bound. It is worth exploring sharper evaluation bounds or new methods that does not need such a bound.

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The appendix is omitted in this abstract.