An Algorithm Computing the Regular Formal Solutions of a System of Linear Differential Equations

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We propose a method for computing the regular singular formal solutions of a linear differential system in the neighbourhood of a singular point. This algorithm avoids the use of cyclic vectors and has been implemented† in the computer algebra system Maple.

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1. Introduction

In this paper, we consider the first-order linear differential system

\[
\frac{dY}{dx} = A(x)Y,
\]

where \( x \) is a complex variable and \( A(x) \) a square matrix of dimension \( n \) the entries of which are formal meromorphic power series. Write

\[
A = x^{-q}(A_0 + A_1x + \cdots) \quad (A_0 \neq 0)
\]

for the series expansion of \( A \), where the coefficients are matrices over a subfield \( K \) of the field of complex numbers. There exists a basis of \( n \) formal solutions of the form (see, e.g. Turritin, 1955; Wasow, 1967)

\[
y_i(t) = e^{q_i(t)}t^{r_i}z_i(t) \quad (i = 1, \ldots, n),
\]

where \( t'^r = x \) for positive integers \( r \), \( q_i \in t^{-1}\overline{K}[t^{-1}], \lambda_i \in \overline{K} \) and \( z_i \in \overline{K}[[t]][\log(t)] \).

Here, \( \overline{K} \) denotes the algebraic closure of \( K \). These solutions form the columns of a formal fundamental matrix solution of (1.1) which can be written as

\[
U(t) = H(t)t^{\Lambda}e^{Q(t)}
\]

with \( t'^r = x \) for a positive integer \( r \), \( H \in \mathcal{M}_n\overline{K}[[t]] \) is a formal matrix power series, \( \Lambda \in \mathcal{M}_n\overline{K} \) is a constant matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \) and \( Q = \text{diag}(q_1, \ldots, q_n) \).

The structure of the formal solutions depends on the nature of the origin as a singular point of the system. If \( Q = 0 \), then \( r = 1 \) and the system is called regular singular, otherwise irregular singular. In this latter case, one has necessarily \( q > 0 \). If \( q = 0 \), the singularity (or the system resp.) is said to be of the first kind (Wasow, 1967) or simple (Hartman, 1964). This is a sufficient condition for a regular singularity, and there is a standard method for the construction of the solutions in this case. An algorithmic

†The program is contained in the package ISOLDE at http://www-lmc.imag.fr/CF/logiciel.html
presentation of this can be found in the last chapter of Sommeling (1993). If \( q > 0 \), it is still possible that the system is regular singular. In that case it is well known (see, e.g. Wasow, 1967) that there then exists a matrix \( T \in \text{GL}_n(K((x))) \) such that the transformation \( Y = TZ \) leads to a new system \( xZ' = BZ \), which has a singularity of the first kind. Here, \( B \) and \( A \) are related by

\[
B = T^{-1}(AT - xT').
\]

Systems which result from such a change of unknown functions are called equivalent. Moser (1960) gave a method which computes for a given system (1.1) a transformation \( T \) that leads to a system with minimal pole order among all equivalent systems. Hence, if the system is regular singular, this yields a system of the first kind, and the classical method can be applied. In Hilali (1987), Moser’s method is improved and there exist several implementations of this algorithm (Sommeling, 1993; Barkatou, 1995). This completely solves the problem of computing the formal solutions in the regular singular case.

In the irregular singular case, it can still happen that there exist solutions of the form

\[
y(x) = x^\lambda z(x),
\]

(1.4)

where \( \lambda \in \bar{K} \) and \( z \in \bar{K}[\![x]\!]^n[\log(x)] \). These solutions correspond to solutions of the form (1.2) with \( q_i = 0 \) (in this case one has \( r_i = 1 \)). We will refer to them as regular formal solutions. The vector space generated by the kinds of solutions will be called the regular formal solution space. Note that its elements have a more general form than in (1.4). In this article, we treat the problem of finding a basis for the regular formal solution space for an arbitrarily given system of the form (1.1). Although the most interesting case is the irregular situation, our method gives a new algorithm for the regular or ordinary case which differs from the classical method.

The problem of computing the regular formal solutions in the irregular singular situation has already been treated algorithmically in the literature. In Wagenführer (1974) the system is studied directly, without applying transformations. This method leads to the evaluation of large matrices the size of which are multiples of the dimension of the system. This makes an efficient implementation difficult. In Hilali and Wazner (1987), a generalization of the Moser algorithm leads to the so-called super-irreducible forms of linear differential systems. This is used in Hilali (1987) to establish a method equivalent to the classical Frobenius method for scalar nth-order equations to the system case. This gives a constructive method treating the irregular singular case. From an algorithmic viewpoint, this method would require a great amount of symbolic manipulations, and it is not clear whether it would be efficient. We do not know about an implementation of this method.

This paper is organized as follows: in Section 2 we consider the subclass of regular formal solutions which do not involve logarithmic terms. We introduce the notion of simple systems. A more detailed presentation and the link to super-irreducible forms can be found in the Appendix. We then present a method for this class of solutions which is a generalization of the main algorithm in Barkatou (1998). Section 3 contains two parts. The first part gives a new method to complete the so-far-computed regular formal solution space. The main idea is to reduce the computation of the logarithmic terms to solving inhomogeneous differential systems which can be done using the method of Section 2. The second part is of theoretical interest. Our main theorem (Theorem 3.2) is not new and can be shown in several ways. It has been presented in Hilali (1987, Chapter 9). Our proof is independent from the results therein. Using the theory of regular matrix pencils
we are able to generalize the demonstration for the case of a singularity of the first kind. This allows us to view simple systems as a natural generalization of this case.

We write \( K((x)) \) for the field of formal meromorphic power series in the variable \( x \). For \( y \in K((x))^n \) we define the valuation \( v(y) \) as the order of the pole of \( y \) at 0. Furthermore, we set \( v(0) = +\infty \). The vector \( \text{lc}(y) \) is the coefficient of the lowest-order term of \( y \) and is called the leading vector of \( y \). For a matrix \( A \in M_n(K) \), we write \( \langle A \rangle \) for the smallest \( K \)-vector space containing the columns of \( A \).

2. Logarithm-free Regular Formal Solutions

We start with the task of computing a subclass of the regular formal solutions, namely the formal meromorphic power series solutions which do not contain logarithmic terms. Consider the following:

**Problem 2.1.** Given the system (1.1), find a non-zero solution of the form

\[
y(x) = x^\lambda \sum_{i=0}^{\infty} g_i x^i \quad (g_0 \neq 0),
\]

where \( \lambda \in \hat{K} \), \( g_i \in \hat{K}^n \), or decide that there is no such solution.

**Remark 2.1.** The solution series of the form (2.1) to be found in Problem 2.1 do, in general, not converge.

In the case of scalar linear differential equations, it is well known that the possible values for \( \lambda \) are roots of the so-called indicial polynomial (Ince, 1944). This polynomial can be computed directly from the coefficients of the equation.

In the system case, the situation is more difficult. It is possible to convert the system into an equivalent scalar linear differential equation and use this to compute the indicial polynomial. We want to avoid this, because this conversion can be costly, especially for large matrices. We will see that one gets more insight into the problem if one considers systems of the more general form

\[
D\theta(Y) = NY,
\]

where \( \theta = x \frac{d}{dx} \), \( D \) and \( N \) are formal power series matrices without a pole, and \( D \) is invertible in \( M_n(K((x))) \). Note that the matrix \( D^{-1} \) may have a pole.

The corresponding matrix differential operator is

\[
\mathcal{D} = D\theta - N.
\]

Denote by \( D_i \) and \( N_i \) the coefficients of the series expansion of \( D \) and \( N \). Inserting an expression of the form (2.1) in (2.2) and comparing coefficients yields the equations

\[
(\lambda D_0 - N_0)g_0 = 0
\]

and

\[
((\lambda + i)D_0 - N_0)g_i = -b_i
\]

where

\[
b_i = ((\lambda + i - 1)D_1 - N_1)g_{i-1} + \cdots + (\lambda D_i - N_i)g_0 \quad (i \geq 1).
\]
Equation (2.4) shows that a necessary and sufficient condition to find a non-zero vector $g_0$ is that there exists a $\mu \in \bar{K}$ such that the matrix $N_0 - \mu D_0$ be a singular matrix. One then may choose $\lambda = \mu$, $g_0 \in \ker(N_0 - \mu D_0)$ and has to successively solve equation (2.5) for increasing values of $i$. These considerations show the following

**Proposition 2.1.** Depending on the structure of the matrix $N_0 - \mu D_0$, one has:

(i) If the matrix $N_0 - \mu D_0$ is invertible for all $\mu \in \bar{K}$, then there exist no solution of the form (2.1).

(ii) If there is a $\mu \in \bar{K}$ such that $N_0 - \mu D_0$ is a singular matrix and the matrices $N_0 - (\mu + i)D_0$ are invertible for all $i \in \mathbb{N}^*$, there is at least one solution of the form

$$y(x) = x^\mu \sum_{i=0}^{\infty} g_i x^i.$$

More precisely, there exist as many independent solutions of this form as there are independent solutions $g_0$ of equation (2.4) with $\lambda = \mu$.

This discussion shows that the matrix $N_0 - \lambda D_0$ plays an important role. If $N_0 - \mu D_0$ is singular for all $\mu \in \bar{K}$, it is not a priori possible to decide the existence of a solution of type (2.1). It is therefore convenient to make the following:

**Definition 2.1.** A differential system (or differential operator, resp.) of the form (2.3) is called simple if the polynomial $P(\lambda) = \det(N_0 - \lambda D_0)$ does not vanish identically in $\lambda$, i.e. $P(\lambda) \neq 0$. In this case, $P$ is called the indicial polynomial of the system (or differential operator resp.).

Hence, if a system is simple, it is immediately possible to give a partial answer to Problem 2.1: there is at least one solution of the required type iff the indicial polynomial $P$ has a degree greater than 0 (since a non-constant polynomial $P$ has always a root $\mu \in \bar{K}$ such that $P(\mu + i) \neq 0$ for all $i \in \mathbb{N}$). In order to compute these solutions, more work has to be done.

**Remark 2.2.** The case of a singularity of the first kind (simple singularity) is a special case of this concept: one can take $D = I$ and $N = A$. The indicial polynomial is then the characteristic polynomial of the matrix $A_0$ (see, e.g. Hartman, 1964).

Given an arbitrary system of the form (1.1), we shall now indicate how one can rewrite it in order to get a system of the form (2.2). Denote by $R_i$ the $i$th row of the matrix $A$, put $\alpha_i = \min(0, v(R_i))$ and

$$D^{-1} = \text{diag}(x^{\alpha_1}, \ldots, x^{\alpha_n}).$$

We then have

$$A = D^{-1} N,$$

where $N = N_0 + N_1 x + \cdots$ is a formal matrix power series and $D = D_0 + D_1 x + \cdots + D_p x^p$ ($p = \max(0, q)$) is an invertible diagonal matrix polynomial. Multiplying by $D$ from the left, the system can be rewritten as in (2.2) where the matrix $D$ is a diagonal matrix.

Of course this does not, in general, mean that the system is simple, i.e. $\det(N_0 - \lambda D_0) \neq 0$. 
Example 1. Consider the system
\[
\theta Y = \begin{pmatrix} x^{-1} + 1 & -x^{-1} & 6x^{-1} \\ 0 & -4 & 2 \\ 3 & 5x^{-2} - 3x^{-1} & 1 \end{pmatrix} Y.
\]
The associated values are \( \alpha_1 = -1 \), \( \alpha_2 = 0 \) and \( \alpha_3 = -2 \), hence we find
\[
D = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^2 \end{pmatrix}, \quad N = \begin{pmatrix} 1 + x & -1 & 6 \\ 0 & -4 & 2 \\ 3x^2 & 5 - 3x & x^2 \end{pmatrix}.
\]
We compute
\[
P(\lambda) = \det(N_0 - \lambda D_0) = \det \begin{pmatrix} 1 & -1 & 6 \\ 0 & -4 - \lambda & 2 \\ 0 & 5 & 0 \end{pmatrix} = -10 \neq 0
\]
and hence this system is simple. Since \( \deg(P) = 0 \), the matrix \( N_0 - \mu D_0 \) is invertible for all \( \mu \in \bar{K} \) and the system has no solution of the form (2.1).

The following is now important.

Proposition 2.2. For an arbitrary system of the form (1.1) there exists an invertible matrix polynomial \( T \in M_n(K[[x]]) \) which transforms the system into an equivalent simple system. Furthermore, there is an algorithm which computes the matrix \( T \) and the corresponding simple system.

We refer to Appendix A for the proof of this proposition.

In the following, we will give a solution to Problem 2.1 for a system which is simple, hence this problem can be answered for an arbitrary system by transforming it into a simple one. This transformation does not change the type of the searched solutions. In the rest of the paper we will assume (unless it is stated differently) that the considered system is simple.

We now want to give a more efficient way for solving Problem 2.1 than to solve the recurrence equations (2.5). We formulate this as the following:

Problem 2.2. Given a simple system of the form (2.2), find all solutions of the form
\[
g(x) = x^\lambda \sum_{i=0}^{\infty} g_i x^i \quad (g_0 \neq 0),
\]
where \( \lambda \in \bar{K}, \ g_i \in \bar{K}^n \).

We first remark that it is always allowed to suppose \( \lambda \in \mathbb{Z} \). Indeed, for a fixed \( \mu \in \bar{K} \) the change of unknown
\[
y = x^\mu z
\]
yields a new operator
\[
\tilde{D} = D\theta - (N - \mu D)
\]
which is still simple. Its indicial polynomial is \( \tilde{P}(\lambda) = P(\lambda + \mu) \). If one chooses \( \mu \) such that \( P(\mu) = 0 \), then \( \tilde{P}(0) = 0 \) and the roots of \( P \) differing by integers from \( \mu \) correspond
Lemma 2.1. Let $\mathcal{D}$ be a simple matrix differential operator and $P$ its indicial polynomial.

(i) Then, for all $y \in K((x))^n$ one has $v(\mathcal{D}(y)) \geq v(y)$ and equality holds iff $lc(y) \not\in \ker ((v(y)D_0 - N_0)$. This is the case, for instance, if $y \neq 0$ and $v(y)$ is not a root of $P$.

(ii) In particular, if $y \in K((x))^n$ is a solution of the equation $\mathcal{D}(y) = b$, then $v(y) \leq v(b)$. If $v(y) < v(b)$, then $P(v(y)) = 0$ and $lc(y) \in \ker ((v(y)D_0 - N_0)$. If $v(y) = v(b)$, then $lc(b) \in \{v(y)D_0 - N_0\}$.

Proof. Write $y = lc(y)x^{v(y)} + \cdots$, where the dots stand for terms of higher valuation. Using the linearity of $\mathcal{D}$, we get

$$\mathcal{D}(y) = \{v(y)D_0 - N_0\}lc(y) \cdot x^{v(y)} + \cdots.$$ 

Hence, it follows that $v(\mathcal{D}(y)) \geq v(y)$ and equality holds iff $(v(y)D_0 - N_0)lc(y) = 0$. Now, if $\mathcal{D}(y) = b$, then $v(b) = v(\mathcal{D}(y)) \geq v(y)$. If $v(y) < v(b)$, then we must have $(v(y)D_0 - N_0)lc(y) = 0$ with $lc(y) \neq 0$ and therefore by definition of the indicial polynomial $P(v(y)) = 0$. If $v(y) = v(b)$, comparing leading coefficients yields

$$(v(y)D_0 - N_0)lc(y) = lc(b),$$

which proves the last part of the lemma. □

If equation (2.10) is solvable in $K((x))^n$, its general formal meromorphic power series solution can be written as

$$y = f_0 + \sum_{j=1}^{m} \beta_j f_j$$

with $f_j \in K((x))^n$, the $\beta_j$ are arbitrary constants, $f_0$ is a particular solution of equation (2.10) and the $f_j$ with $j > 0$ form a basis of the solution space of the homogeneous problem $\mathcal{D}(y) = 0$. Following Barkatou (1998), we will now explain how to compute this solution up to the order $k$ for a given integer $k$. This means that we are only interested in the terms that have a valuation less than $k$. This method will solve the more general
problem where the right-hand side \( b \) depends linearly on parameters, i.e. \( b \) is of the form

\[
b = b_0 + \sum_{j=1}^{l} p_j b_j,
\]

where the \( b_j \in K((x))^n \) and the \( p_j \) are given parameters. Here, solving \( \mathfrak{D}(y) = b \) has the same meaning as in Singer (1991, Section 3, p. 262). More precisely, we will find \( y_0, \ldots, y_m \in K((x))^n \) and a system \( \mathcal{L} \) of linear equations in \( l+m \) variables with coefficients in \( K \) such that \( \mathfrak{D}(y) = b \) iff

\[
y = y_0 + c_1 y_1 + \cdots + c_m y_m,
\]

where \( c_i \in K \) and \( p_1, \ldots, p_k, c_1, \ldots, c_m \) satisfy \( \mathcal{L} \). For example, consider the system of dimension \( n = 1 \) given by the differential equation \( xy' - y = p_1 x \). One has

\[
y = c_1 x, \quad c_1 \in K, \quad \mathcal{L} = \{ p_1 = 0 \}.
\]

For the equation \( xy' - y = x^3 + p_1 x^3 + p_2 \), one finds

\[
y = \frac{1}{2} x^3 + c_1 + c_2 x + c_3 x^3, \quad c_1, c_2, c_3 \in K, \quad \mathcal{L} = \{ c_1 + p_2 = 0, c_3 - \frac{1}{2} p_1 = 0 \}.
\]

Let us denote by \( \mathcal{P} = \{ p_1, \ldots, p_k \} \) the set of parameters (which may be empty). If \( b \neq 0 \), we set \( \delta = v(b) \) and \( b = lc(b)x^\delta + \cdots \). If \( b = 0 \), let \( \delta = +\infty \) and \( lc(b) = 0 \). Note that the components of \( lc(b) \) are linear in the parameters \( p_j \). Write

\[
y = cx^\mu + z \quad (2.11)
\]

with \( 0 \neq c \in K^n, \mu \in \mathbb{Z} \) and \( \mu < k \). The problem is to find a vector \( c \in K^n \) and an integer \( \mu < k \) such that \( \mathfrak{D}(y) = b \) and \( v(z) > \mu \). One has

\[
\mathfrak{D}(z) = b - \mathfrak{D}(cx^\mu) = \sum_{j=1}^{l} p_j b_j x^{\mu + j} - (\mu D_0 - N_0) c \cdot x^\mu + \cdots. \quad (2.12)
\]

From Lemma 2.1, we know that \( v(\mathfrak{D}(z)) \geq v(z) \) and hence a necessary condition that \( \mu \) and \( c \) exist is that the valuation of the right-hand side of (2.12) must be \( > \mu \). This last condition holds only if one chooses \( \mu < \delta \) as an integer root of the indicial polynomial and \( c \), such that \( (\mu D_0 - N_0) c = 0 \) or \( \mu = \delta \) and \( c \) a solution of the linear equation \( lc(b) - (\delta D_0 - N_0) c = 0 \). There are several possibilities: let \( \mathcal{R} \) be the set of the integer roots of the indicial polynomial \( P \) which are smaller than \( k \) (note that \( \mathcal{R} \) may be empty).

1. If \( \mathcal{R} = \emptyset \) and \( \delta \geq k \), then the valuation of the right-hand side of (2.12) is equal to \( \mu \) for all integers \( \mu < k \). So, in this case there is no possible couple \( (\mu, c) \).

2. If there is a \( \mu \in \mathcal{R} \) with \( \mu < \delta \), one can choose \( \mu \) as this root and \( c \) an arbitrary element of \( \ker(\mu D_0 - N_0) \). In the algorithm, we will take in this case \( \mu = \min \mathcal{R} \).

3. If all elements of \( \mathcal{R} \) are greater than or equal to the valuation of the right-hand side of (2.12), then the only possible choice of \( \mu \) is \( \mu = \delta \). This leads to two cases:

   a. If one can determine the parameters \( p_j \) in such a way that \( lc(b) \) belongs to \( (\delta D_0 - N_0) \) then one can set \( \mu = \delta \) and \( c \) any solution of the equation \( lc(b) - (\delta D_0 - N_0) c = 0 \). For instance, if \( P(\delta) \neq 0 \) then \( lc(b) \in (\delta D_0 - N_0) \) for all values of the \( p_j \) and in this case \( c \) is unique: \( c = (\delta D_0 - N_0)^{-1} lc(b) \).

   b. If \( lc(b) \notin (\delta D_0 - N_0) \) for all choice of the \( p_j \), then there is no possible couple \( (\mu, c) \).
This discussion leads to a recursive algorithm. The idea is the following: after having found possible monomials $cx^\mu$ as in (2.11), one performs the substitution $y = cx^\mu + z$. This gives a new equation in $z$, and one continues as before, only considering monomials of valuation $> \mu$. One starts with the sets $\mathcal{R}$ and $\mathcal{P}$ as defined. During the execution of the algorithm, these sets change. At one stage of recursion:

— the set $\mathcal{P}$ stores all parameters which occur in the currently computed terms,
— the elements of $\mathcal{R}$ have the following property: they are the integer roots of the indicial polynomial $P$ which are $< k$ and have not yet been used in the so-far-computed terms.

There are two situations in which the set $\mathcal{P}$ changes. At step 2, we compute a basis of $\ker(\mu D_0 - N_0)$, and new parameters are introduced. In step 3 (a), the condition $\text{lc}(b) \in \langle \delta D_0 - N_0 \rangle$ entails linear equations on the $p_i$ which are obtained by Gaussian elimination. We substitute for these conditions in the so-far-computed terms, this may eliminate some elements of $\mathcal{P}$. Note that the solutions of the equation $(\mu D_0 - N_0)c = \text{lc}(b)$ can add new parameters to $\mathcal{P}$.

The set $\mathcal{R}$ will be modified in the following way: an element is taken away whenever it is used as a $\mu$ in (2.11). This method is a generalization of the method in Barkatou (1998) for computing polynomial solutions. We refer to this paper for a slightly different description of how to organize this process. Note that one can also apply this idea for the scalar case in order to get an algorithm somehow similar to Abramov and Kvasenko (1991).

**Example 2.** Let us study the system

$$\mathcal{D}(Y) = \begin{pmatrix} x^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \theta(Y) - \begin{pmatrix} 1-x^2 & x^2 & 1+x+x^3 \\ x^2 & 0 & 1+x^3 \\ x^3 & -x & 1 \end{pmatrix} Y = 0.$$ 

We want to find a formal meromorphic power solution series up to order $k = 3$. The associated indicial equation is $P(\lambda) = \det(N_0 - \lambda D_0) = \lambda(\lambda - 1)$. The list of integer roots is $\mathcal{R} = \{0, 1\}$. We start with a zero right-hand side $b = 0$ which gives $\delta = +\infty$. Hence, we choose $\mu = \min \mathcal{R} = 0$ and $c \in \ker(N_0)$. We obtain the vector $c = (0, p_0, 0)^t$ ($p_0$ arbitrary $\in \mathcal{K}$) and substitute $y = c + z$. The new right-hand side becomes $b = 0 - \mathcal{D}(c) = (p_0x^2, 0, -p_0x)^t$ and we have $\mathcal{R} = \{1\}$, $\delta = v(b) = 1$ and $\text{lc}(b) = (0, 0, -p_0)^t$.

So we choose $\mu = 1$ and must solve the linear system

$$(D_0 - N_0)c = \text{lc}(b) \iff \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \cdot c = \begin{pmatrix} 0 \\ 0 \\ -p_0 \end{pmatrix}.$$ 

This equation has a solution iff $p_0 = 0$. We substitute $p_0 = 0$ in the so-far-computed solution and right-hand side and obtain $y = b = 0$. A solution of the above system is then $c = (p_1, -p_1, -p_1)^t$. The new solution becomes $y = (p_1, x, -p_1 x, -p_1 x)^t$ and $\mathcal{R}$ is now the empty set. The right-hand side is $0 - \mathcal{D}(cx) = (-p_1, 0, p_1)^t x^2 + \cdots$ and the next monomial is determined uniquely by $c = (2D_0 - N_0)^{-1}(-p_1, 0, p_1)^t = (0, \frac{1}{2} p_1, p_1)^t$. One
has $D(y) - b \in O(x^3)$ and thus the solution up to order 3 is
\[ y(x) = p_1 \cdot \begin{pmatrix} x \\ -x + \frac{1}{2}x^2 \\ -x + x^2 \end{pmatrix}. \]

The number of independent formal series solutions computed by our algorithm depends on the value of $k$. A solution $y$ with $v(y) \geq k$ will not appear in the output of the general solution. On the other hand, it is possible that a computed solution $y \in K[x,x^{-1}]^n$ cannot be extended to an infinite power series solution. The reason is that $y$ may be only a particular portion of some more general solution. In the previous example, if one had computed the solutions up to order $k = 1$, one would have obtained the vector $(0, p_0, 0)^t$ which does not correspond to a solution in $K((x))^n$.

**Example 3.** The system
\[ \theta Y = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} Y + \begin{pmatrix} 0 \\ x^2 \end{pmatrix} \]
has no infinite power series solution. However, the vector $y = (p_1x, 0)^t$ which will be computed by our algorithm for $k = 2$ is a solution of the system up to order $O(x^2)$.

If one chooses $k$ greater than the largest integer root of $P$, the matrices $\delta D_0 - N_0$ will be invertible for all $\delta > k$. This assures that for the so-far-computed solutions an arbitrary number of further terms can be computed.

### 3. Logarithmic Terms

In the previous section, we reviewed a method for efficiently computing the formal meromorphic power series solutions of the form (2.1). It is well known that if one wants to complete the full solution space of regular formal solutions, one has to consider a more general class involving powers of logarithms (see, e.g. Wasow, 1967). In the first part of this section, we will show how to compute the full set of independent regular formal solutions. This method is easy to implement and we will illustrate it by an example. The second part has theoretical interest and is added for reasons of completeness, although the main result (Theorem 3.2) is not new. Our main tool is the concept of regular matrix pencils and this allows us to give a proof quite similar to the techniques used in the case of a singularity of the first kind. We remark that our algorithm does not need these concepts which are only introduced for theoretical purpose.

#### 3.1. the algorithm

In this section, we let $D$ be a simple operator and denote by $P$ its indicial polynomial. A regular formal solution is a linear combination of solutions of the form
\[ y_k(x) = x^\lambda \left( h_s(x) + \log(x)h_{s-1}(x) + \cdots + \frac{\log^{s-1}(x)}{(s-1)!}h_1(x) \right), \tag{3.1} \]
where $\lambda \in \bar{K}$ and $h_k \in \bar{K}((x))^n$ for $k = 1, \ldots, s$.

By means of a transformation of the form (2.8) we can suppose $\lambda \in \mathbb{Z}$. Hence, computing a basis of the regular formal solution space can be reduced to solving
Problem 3.1. Given the equation $\mathcal{D}(y) = 0$, compute all solutions of the form

$$y_s(x) = h_s(x) + \log(x)h_{s-1}(x) + \cdots + \frac{\log^{s-1}(x)}{(s-1)!}h_1(x), \quad (3.2)$$

where the $h_i \in \bar{K}(x)_n$.

The following lemma reduces this problem to finding logarithm-free formal solutions of non-homogeneous differential equations.

Lemma 3.1. The equation $\mathcal{D}(y) = 0$ has a solution of the form $(3.2)$ iff $\mathcal{D}(h_1) = 0$ and $\mathcal{D}(h_k) + Dh_{k-1} = 0$ for $2 \leq k \leq s$.

Proof. A direct computation gives

$$\mathcal{D}(y_s) = \sum_{k=1}^{s} \frac{1}{(s-k)!} \mathcal{D}(h_k \log^{s-k}(x))$$

$$= \sum_{k=1}^{s-1} \frac{1}{(s-k)!} (\mathcal{D}(h_k) \log^{s-k}(x) + (s-k)Dh_k \log^{s-k-1}(x)) + \mathcal{D}(h_s)$$

$$= \frac{1}{(s-1)!} \mathcal{D}(h_1) \log^{s-1}(x) + \sum_{k=2}^{s} \frac{1}{(s-k)!} (\mathcal{D}(h_k) + Dh_{k-1}) \log^{s-k}(x).$$

This expression is a polynomial in $\log(x)$ and is zero iff its coefficients vanish. This proves the lemma. □

This shows that if $y_s$ is a solution as in $(3.2)$, then so are $y_1, \ldots, y_{s-1}$ where

$$y_1(x) = h_1(x),$$

$$y_2(x) = h_2(x) + \log(x)h_1(x),$$

$$\vdots$$

$$y_{s-1}(x) = h_{s-1}(x) + \log(x)h_{s-2}(x) + \cdots + \frac{\log^{s-2}(x)}{(s-2)!}h_1(x).$$

Since $y_1 = h_1$ is a solution of the form $(2.1)$, we know that $v(h_1)$ is a root of the indicial polynomial. This means that we can find all possible values for $\lambda$ in $(3.1)$ (modulo an integer) by considering the roots of the indicial polynomial $P$.

Using the method of Section 2, we proceed as follows in order to find solutions with an $s$ maximal:

1. Solve the homogeneous problem $\mathcal{D}(h_1) = 0$ ($h_1 \neq 0$). This gives a general solution containing parameters. We set $k = 1$.
2. Let $k = k + 1$.
3. Solve the equation $\mathcal{D}(h_k) = -Dh_{k-1}$. The algorithm of the previous section will either find a parametrized solution or decide that the condition $h_1 \neq 0$ will lead to no such solution.
4. If this equation has no solution, we have $s = k - 1$ and we return the general solution $h_s + \log(x)h_{s-1} + \cdots + \frac{1}{(s-1)!} \log^{s-1}(x)h_1$. Otherwise, go to step 2.
Thus, the value for $s$ is computed in $s + 1$ steps. In the next section, we will show that $s$ is bounded by the number of integer roots of $P(\lambda)$ counted with multiplicities, which is bounded by $n$. So this algorithm terminates after at most $n$ steps. From the above remarks follows

**Theorem 3.1.** If one applies this method for all roots of the indicial equation (by choosing representatives for the roots differing by integers), one obtains a basis of the regular formal solution space.

**Example 4.** Let us again consider the example from Section 2. We have found a one-dimensional solution space of logarithm-free regular formal solutions. We now will see that the regular formal solution space is of dimension two, and we will compute it. Denote by $h_1$ the solution found previously. We have to solve

$$\mathfrak{D}(h_2) = -Dh_1 = \begin{pmatrix} -p_1 x^3 \\ p_1 (x - \frac{1}{2} x^2) \\ p_1 (x - x^2) \end{pmatrix}.$$  

The first step of the algorithm is the same as in the homogeneous case, since $\mathcal{R} = \{0, 1\}$, $\delta = 1$ and $\min \mathcal{R} = 0 < \delta$. We obtain again the parametrized vector $c = (0, p_1, 0)^t$. The new right-hand side is then $-Dh_1 - \mathfrak{D}(c) = (0, p_1, p_1 - p_2)^t x + \cdots$, hence $\delta = 1$. The equation

$$(D_0 - N_0) c = lc(b) \iff \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \cdot c = \begin{pmatrix} 0 \\ p_1 \\ p_1 - p_2 \end{pmatrix}$$

has a solution iff $p_1 - p_2 = 0$. This gives the set of conditions $\mathcal{C} = \{p_1 = p_2\}$ and yields the solution $c = (-p_3, p_3 + p_2, p_3)^t$. Note that this has introduced a new parameter. Computing the new right-hand side, one finds $\delta = 2$ and $lc(b) = (p_2 + p_3, -\frac{1}{2} p_2, -2 p_2 - p_3)^t$. The next term is uniquely determined by $(D_0 - N_0)^{-1} lc(b)$. We finally find

$$h_2 = \begin{pmatrix} -p_3 x + p_2 x^2 \\ p_2 + (p_2 + p_3)x - (\frac{5}{2} p_2 + \frac{1}{2} p_3)x^2 \\ p_3 x - (2p_2 + p_3)x^2 \end{pmatrix}.$$  

We substitute $p_1 = p_2$ in $h_1$ and get the general regular formal solution

$$y_2(x) = \begin{pmatrix} \log(x)(-p_2 x - p_3 x + p_2 x^2) \\ \log(x)(-p_2 x + p_2 x^2) + (p_2 + p_3)x - (\frac{5}{2} p_2 + \frac{1}{2} p_3)x^2 \end{pmatrix}.$$  

One has $\mathfrak{D}(y_2) \in O(x^4)$. So far, we have found two independent regular solutions and therefore $s \geq 2$. The equation $D(h_3) = -Dh_2$ leads to the linear system

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \cdot c = \begin{pmatrix} 0 \\ -p_2 \\ 0 \end{pmatrix}.$$  

The condition for solvability is $\{p_2 = 0\}$. But this is in contradiction with $h_1 \neq 0$. Indeed, if we substitute $p_2 = 0$ in $h_1$, we get $h_1 = 0$. This means that the term $\frac{1}{2} \log^2(x)h_1$ in the general solution would vanish. Hence $s = 2$ is maximal. Since we have treated all roots of the indicial polynomial, we have also found all regular formal solutions of the system.
3.2. THE DIMENSION OF THE REGULAR FORMAL SOLUTION SPACE

The goal of this section is to establish links between the structure of the regular formal solutions and the properties of the matrix pencil $N_0 - \lambda D_0$. In particular, we will obtain (in a different way as in Hilali, 1987, Chapter 9) the following:

**Theorem 3.2.** Let $D$ be a simple operator and $P(\lambda)$ its indicial polynomial. Denote by $\mu$ a root of $P$ and $\mu_1, \ldots, \mu_d$ the roots of $P$ with $\mu - \mu_i \in \mathbb{Z}$. Denote by $m_i$ the multiplicity of the root $\mu_i$. Then there exist precisely $m = m_1 + \cdots + m_d$ independent regular formal solutions of the form

$$x^\mu \left( h_s(x) + \log(x)h_{s-1}(x) + \cdots + \frac{\log^{s-1}(x)}{(s-1)!}h_1(x) \right)$$

with $h_k \in \mathbb{K}(x)$ ($k = 1, \ldots, s$). Moreover, one has $s \leq m$. Hence, the dimension of the regular formal solution space is equal to the degree of $P$.

The proof of this theorem is given by combining Propositions 3.2 and 3.3, and the fact that the space of regular formal solutions has a basis which consists of regular formal solutions of the form (3.1). The latter fact can be seen from the form of the formal fundamental matrix solution (1.3), see also, e.g. Wagenführer (1974).

This result shows that the polynomial $P(\lambda)$ plays the same role as the indicial polynomial in the case of $n$th-order linear differential equations.

**Definition 3.1.** Let $A, B \in \mathcal{M}_n \mathbb{K}$ be two square matrices and $\lambda$ an indeterminate. $A - \lambda B$ is called a regular matrix pencil, if one has $\det(A - \lambda B) \neq 0$.

The theory of regular matrix pencils is well known. We refer to Gantmacher (1966) for more details on the following concepts and the proof of Theorem 3.3.

**Definition 3.2.** Two regular matrix pencils $A - \lambda B$ and $\hat{A} - \lambda \hat{B}$ are called strictly equivalent if there exist matrices $S, T \in \text{GL}_n(\mathbb{K})$ such that

$$S(A - \lambda B)T = \hat{A} - \lambda \hat{B}.$$

The following theorem gives the normal form for the class of strictly equivalent regular matrix pencils.

**Theorem 3.3.** (Weierstrass Normal Form) Any regular matrix pencil is strictly equivalent to a matrix pencil of the form

$$\begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} - \lambda \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix},$$

where these two matrices have the same block structure, the square matrix $H$ is of the form

$$\begin{pmatrix} 0 & * & 0 \\ \vdots & \ddots & * \\ 0 & \cdots & 0 \end{pmatrix} \quad (\ast \in \{0, 1\}) \quad (3.3)$$

and $J$ is a matrix in Jordan normal form.
We will call the roots of the polynomial $\det(A - \lambda B)$ the \textit{eigenvalues} of the regular matrix pencil. For an eigenvalue $\mu \in \bar{K}$ we will call a non-zero vector $e \in \ker(A - \mu B)$ an \textit{eigenvector} w.r.t. the eigenvalue $\mu$. A vector $e$ is called a \textit{generalized eigenvector}, if there exist a (generalized) eigenvector $f \in K^n$ such that $(A - \mu B)e = f$. Using the above normal form, it is easy to see that a chain of generalized eigenvectors corresponds to a block

$$\begin{pmatrix} \mu & 1 \\ \vdots & \ddots & 1 \\ 0 & \cdots & \mu \end{pmatrix}$$

in the matrix $J$. Let $m$ be the multiplicity of $\mu$ as a root of the polynomial $\det(A - \lambda B)$. Then, there are overall $m$ linearly independent (generalized) eigenvectors associated with $\mu$.

**Definition 3.3.** Two operators $D = D\theta - N$ and $\hat{D} = \hat{D}\theta - \hat{N}$ are said to be equivalent, iff there exist matrices $S, T \in GL_n(\bar{K}(x))$ such that $\hat{D} = SDT$ and $\hat{N} = S(NT - D\theta T)$.

The solutions of two equivalent systems $D(y) = 0$ and $\hat{D}(z) = 0$ as above are related by $y = Tz$.

Using the terminology of eigenvalues and eigenvectors, equation (2.4) shows the following:

**Proposition 3.1.** Let $D$ be a simple operator and $y$ a logarithm-free formal meromorphic power series solution of the form (2.1) with $\lambda = \mu$. Then, $\mu$ is an eigenvalue of the regular matrix pencil $N_0 - \lambda D_0$, the leading vector $g_0$ is an eigenvector w.r.t. $\mu$, and there are at most $d$ independent solutions of this form, where $d$ denotes the dimension of the corresponding eigenspace.

We now consider a particular case:

**Proposition 3.2.** Let $D$ be a simple differential operator and $P(\lambda)$ its indicial polynomial. Let $\mu$ be a root of $P$ with multiplicity $m$. Suppose that for all $0 \neq p \in \mathbb{Z}$ we have $P(\mu + p) \neq 0$. Then there exist $m$ regular formal solutions of the form (3.1) with $\lambda = \mu$.

**Proof.** Without loss of generality we can suppose $\mu \in \mathbb{Z}$. By means of constant matrices $S, T \in M_n(\mathbb{K})$, we can find an equivalent simple system such that the associated regular matrix pencil $N_0 - \lambda D_0$ is in Weierstrass normal form. Let

$$f_1^{(1)}, \ldots, f_d^{(1)}$$
be a basis of the eigenspace $\ker(N_0 - \mu D_0)$. Hence, there are $d$ independent formal solutions of the form

$$x^\mu \sum_{j=0}^{\infty} g_{i,j} x^j$$

with $g_{i,0} = f_i^{(1)}$ ($i = 1, \ldots, d$). Now let $m_i \in \mathbb{N}^*$ be maximal such that there exists a chain of linearly independent vectors

$$f_{i}^{(2)}, \ldots, f_{i}^{(m_i)} \quad (i = 1, \ldots, d)$$
verifying

\[(N_0 - \mu D_0) f_i^{(k)} = f_i^{(k-1)} \quad (3.6)\]

for \( k = 2, \ldots, m_i \). The vectors

\[f_1^{(1)}, \ldots, f_1^{(m_1)}, \ldots, f_d^{(1)}, \ldots, f_d^{(m_d)} \quad (3.7)\]

are linearly independent and form a basis of the generalized eigenspace w.r.t. \( \mu \) which is of dimension \( m = m_1 + \cdots + m_d \). The numbers \( m_i \) correspond to Jordan blocks of the form (3.4) of dimension \( m_i \).

We will now proceed in two steps. We will first construct \( m \) linearly independent regular formal solutions of the form (3.1) with \( \lambda = \mu \) having as leading coefficients the (generalized) eigenvectors (3.7), this shows that there are at least \( m \) such solutions. We then will show that there is in fact a one-to-one correspondence of regular formal solutions of the form (3.1) with \( \lambda = \mu \) and (generalized) eigenvectors associated to \( \mu \).

The following considerations are valid for all \( 1 \leq i \leq d \), and we suppress the index \( i \) for the vectors \( f_i^{(k)} \) in order to simplify notation, if there is no confusion. Set \( h_1 \) a solution as in (3.5) with \( lc(h_1) = f^{(1)} \). We will show that with the chain of vectors (3.6) we can associate a set of \( s = m_i \) regular formal solutions

\[y_1(x) = h_1(x), \quad y_2(x) = h_2(x) + \log(x)h_1(x), \quad \vdots \quad y_{m_i}(x) = h_{m_i}(x) + \log(x)h_{m_i-1}(x) + \cdots + \frac{\log^{m_i-1}(x)}{(m_i-1)!}h_1(x)\]

and \( lc(h_k) = f^{(k)} \) \( (k = 1, \ldots, m_i) \). From Lemma 2.1 we obtain the non-homogeneous linear differential equations

\[\mathfrak{D}(h_k) + Dh_{k-1} = 0 \quad (3.8)\]

for the \( h_k \). Inserting the series expansion

\[h_k = x^\mu \sum_{j=0}^{\infty} h_{k,j} x^j\]

into equation (3.8), we obtain the conditions

\[(\mu D_0 - N_0) h_{k,0} = - D_0 h_{k-1,0} \quad (3.9)\]

for the leading vectors of the \( h_k \). Multiplying equation (3.6) by \( -D_0^{k-1} \) on the left and remarking that \( D_0 \) and \( N_0 \) commute (this follows from the Weierstrass normal form), we obtain

\[(\mu D_0 - N_0) D_0^{k-1} f^{(k)} = - D_0^{k-1} f^{(k-1)} \quad (k = 2, \ldots, m_i) \quad (3.10)\]

and this shows that we obtain a particular solution for (3.9) by setting \( h_{k,0} = D_0^{k-1} f^{(k)} \) \( (k = 2, \ldots, m_i) \). Note that \( h_{k,0} \neq 0 \). This can be seen as follows: first one sees that \( D_0 f^{(1)} \neq 0 \) for all \( i \), because \( D_0 f^{(1)} = 0 \) implies \( (N_0 - (\mu + c) D_0) D_0^{k-1} f^{(1)} = 0 \) for all \( c \in K \). Since the system is simple, it follows that \( D_0^{k-1} f^{(1)} = 0 \). By induction, it follows that \( f^{(1)} = 0 \), a contradiction, so \( D_0 f^{(1)} \neq 0 \). Now from \( D_0 f^{(k)} = 0 \) it follows that \( 0 = (N_0 - \mu D_0) D_0^{k-1} f^{(k)} = D_0^{k-1} f^{(k-1)} \). By induction, it follows that \( D_0^{k-1} f^{(1)} = 0 \), a contradiction, so \( D_0 f^{(k)} \neq 0 \) for all \( k, i \).
We have found the leading vectors of the \( h_k \). Now, the coefficients of higher-order terms are determined uniquely, since \( \det(D_0(\mu + p) - N_0) \neq 0 \) for \( p \in \mathbb{N}^* \). The resulting solutions are independent due to the increasing powers of logarithms. This shows that there are at least \( m = m_1 + \cdots + m_d \) regular formal solutions associated with \( \mu \).

In the second step, we will see that there are at most \( m \) linearly independent regular formal solutions of the form (3.1) with \( \lambda = \mu \). As we have seen in Section 3.1, if

\[
y_s(x) = h_s(x) + \log(x)h_{s-1}(x) + \cdots + \frac{\log^{s-1}(x)}{(s-1)!}h_1(x)
\]

is a solution, we have \( \mathcal{D}(h_1) = 0 \) and \( \mathcal{D}(h_k) + Dh_{k-1} = 0 \). This gives the conditions

\[
(N_0 - \mu D_0)lc(h_1) = 0
\]

and

\[
(N_0 - \mu D_0)lc(h_k) = D_0 lc(h_{k-1}) \quad (k = 2, \ldots, s)
\]

for the leading coefficients of the \( h_k \). Taking into account the special form

\[
N_0 - \mu D_0 = \begin{pmatrix}
I - \mu H & 0 \\
0 & J - \mu I
\end{pmatrix}
\]

and carrying out equations (3.12) and (3.13) per blocks one sees that the first \( n - \deg(P) \) entries of the \( lc(h_k) \) must be zero. The \( lc(h_k) \) are of the form

\[
\begin{pmatrix}
0 \\
g^{(k)}
\end{pmatrix}
\]

with \( g^{(k)} \in \mathbb{K}^{\deg(P)} \). Hence, the \( g^{(k)} \) are (generalized) eigenvectors of the matrix \( J \) w.r.t. the eigenvalue \( \mu \). Hence, the number of independent regular formal solutions of the form (3.11) is bounded by the dimension of the generalized eigenspace which is \( m \). This proves the proposition. \( \square \)

We obtain:

**Corollary 3.1.** If \( \mathcal{D} \) is a simple operator and the roots of its indicial polynomial do not differ by integers, there are \( \deg(P) \) linearly independent regular formal solutions of the form (3.11).

We now will show that it is always possible to reduce to this case.

**Proposition 3.3.** Let \( \mathcal{D} \) be a simple operator and \( P \) its indicial polynomial. Then, there exist matrices \( S, T \in GL_n(\mathbb{K}(\!(x)\!)) \) which transform \( \mathcal{D} \) into an equivalent simple operator \( \tilde{\mathcal{D}} \) with indicial polynomial \( \tilde{P} \), such that for two roots \( \mu_i, \mu_j \) of \( \tilde{P} \) one has either \( \mu_i = \mu_j \) or \( \mu_i - \mu_j \notin \mathbb{Z} \).

**Proof.** We can assume that the associated regular matrix pencil \( N_0 - \lambda D_0 \) is in Weierstrass normal form. Let \( \mu \) be a root of \( P \) and write

\[
N_0 = \begin{pmatrix}
I_{n_1} & 0 & 0 \\
0 & J_1 & 0 \\
0 & 0 & J_2
\end{pmatrix}
\]
and
\[
D_0 = \begin{pmatrix}
H_{n_1} & 0 & 0 \\
0 & I_{n_2} & 0 \\
0 & 0 & I_{n_3}
\end{pmatrix},
\]
where \( n = n_1 + n_2 + n_3 \), \( H_{n_1} \in \mathcal{M}_{n_1}(\bar{K}) \) is nilpotent of the form (3.3), \( J_1 \in \mathcal{M}_{n_2}(\bar{K}) \) and \( J_2 \in \mathcal{M}_{n_3}(\bar{K}) \) are in Jordan form. Furthermore, we choose \( J_1 \) such that it has a unique eigenvalue \( \mu \). Now consider the transformation
\[
U = \text{diag}(I_{n_1}, xI_{n_2}, I_{n_3})
\]
and study the new equivalent system which results from a transformation with \( S = U^{-1} \) and \( T = U \). It is easily verified that for the new system
\[
\hat{N}_0 = \begin{pmatrix}
I_{n_1} & 0 & 0 \\
* & J_1 - I_{n_2} & * \\
0 & 0 & J_2
\end{pmatrix}, \quad \hat{D}_0 = \begin{pmatrix}
H_{n_1} & 0 & 0 \\
* & I_{n_2} & * \\
0 & 0 & I_{n_3}
\end{pmatrix}.
\]
One sees that this transformation changes the eigenvalue \( \mu \) to \( \mu - 1 \) and does not affect the other eigenvalues different from \( \mu \). Hence, by repeated use of transformations of the above type (the system must be brought into normal form after each transformation), we obtain the result. \( \Box \)

REMARK 3.1. If the singularity is of the first kind, these concepts become the well-known classical results. One then has \( D = I, N = A \), and the associated matrix pencil is \( A_0 - \lambda I \). Our proof is then similar to the proof given in Hartman (1964, Chapter IV, Theorem 11.4). One can derive an algorithm from this proof, but we do not use this for our method.

4. Conclusion

In this paper, we have resumed a method for computing the regular formal solutions of a linear system of differential equations based on the work in Barkatou (1998) and have given a new algorithm for the logarithmic terms. This is not the only way to solve such systems: other techniques commonly try to convert the system into one or several scalar linear differential equations. The algorithm in Barkatou (1993) gives an heuristic for finding an equivalent companion form block diagonal system. In most cases, this will yield one block and is therefore more or less equivalent to the construction of a cyclic vector. There are algorithms and programs for solving those kind of equations (Barkatou, 1988; van Hoeij, 1996; Pfügel, 1997b). However, this approach may be very costly, especially for large matrices. A good implementation of the super-reduction algorithm of Hilali and Wazner (1987) allows us to treat even big systems. The overall computation time of our algorithms depends on several factors. Super-reduction takes increasingly more time the more the system has to be prepared. This is related to the difference of the actual pole order and the pole order of the resulting super-irreducible matrix. For example, a system (1.1) of dimension \( n = 10 \) and \( q = 0 \) is already super-irreducible, hence one can immediately start with the computation of the regular formal solutions. A cyclic vector computation for a matrix of this size is very costly anyway and it appears unclear how it depends on the structure of the matrix. A system which is regular singular but which has a large \( q \) on the other hand will need a lot of work in the reduction step. Another
factor is of course the demanded accuracy and the presence of the logarithmic factors which cause iterated calls of the algorithm of Section 2.

The problem of computing local formal solutions is also important for finding closed form solutions of linear differential systems. The valuations of the regular formal solutions at the different singularities including infinity give bounds for polynomial or rational solutions, and the logarithm-free solution series computed in Section 2 are candidates for polynomial solutions, as presented in Barkatou (1998). One can find a method for computing the polynomials \( q_i \) in Barkatou (1997). Together with this method, it is possible to compute a complete basis of formal solutions of the form (1.2). A method for computing only the subset of formal solutions of ramification 1 is used in Pfügel (1997a) for finding exponential solutions.

Our results show that it is possible to give efficient algorithms for the symbolic resolution of systems directly, and future applications may benefit from these methods.

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References


Appendix A. Super-irreducible Forms of Linear Differential Systems

In this section, we recall the notion of super-irreducible forms of linear differential systems as introduced in Hilali and Wazner (1987) and show that a super-irreducible form can be transformed to a system which is simple. Consider

\[ x^{q+1}Y' = A(x)Y, \]

where \( A = A_0 + A_1x + \cdots \) and \( q \geq 0 \). Let \( n \) be the dimension of \( A \) and \( n_i \) the number of columns of \( A \) having valuation \( i \). Define for \( 1 \leq k \leq q \) the rational number \( m_k(A) \) as follows: if \( q = 0 \), we set \( m_k(A) = 1 \), otherwise

\[ m_k(A) = q + \frac{n_0}{n} + \frac{n_1}{n^2} + \cdots + \frac{n_{k-1}}{n^k}. \]

Furthermore, let

\[ \mu_k(A) = \min \{ m_k(T^{-1}(AT - x^{q+1}T')) | T \in \text{GL}_n(K((x))) \}. \]

Definition A1. (Hilali and Wazner, 1987) The system (or the matrix \( A \) resp.) is called super-irreducible iff \( m_q(A) = \mu_q(A) \).

By definition of \( m_k \), the condition \( m_q(A) = \mu_q(A) \) implies \( m_k(A) = \mu_k(A) \) for \( 1 \leq k < q \). Define the integer

\[ r_k = kn_0 + (k-1)n_1 + \cdots + n_{k-1} \]

and

\[ \varphi_k(x, \lambda) = x^{r_k} \det \left( \frac{A}{x^k} + \lambda I \right). \]

Then we have

\[ \varphi_k(x, \lambda) \in K[[x]][\lambda]. \]

This can be seen in the following way (we recall the proof from Hilali and Wazner, 1987, for the special case \( k = q \), because it is useful in order to understand better Proposition A1 and the link to simple systems): let \( \alpha = \text{diag}(\alpha_1, \ldots, \alpha_n) \) and \( \alpha_i = \min(0, v(c_i) - q) \) where \( c_i \) denotes the \( i \)th column of \( A \). Then

\[ x^{-\alpha} = D_0 + D_1x + \cdots + D_qx^q \]

and \( N = Ax^{-\alpha} \in \mathcal{M}_n K[[x]] \). One verifies \( \det(x^{-\alpha}) = x^{r_s} \) and

\[ \varphi_q(x, \lambda) = \det(N + \lambda x^{-\alpha}) \in K[[x]][\lambda]. \]

In Hilali and Wazner (1987), the following constructive criterion is given:

Proposition A1. (Hilali and Wazner, 1987) The system is super-irreducible, iff the polynomials

\[ \Theta_k(\lambda) = \varphi_k(0, \lambda) \quad (k = 1, \ldots, q) \]

do not vanish identically in \( \lambda \).

In Hilali and Wazner (1987) an algorithm for the computation of a super-irreducible form is given. They show that an arbitrary system can be transformed into a super-irreducible
one using a polynomial transformation matrix. An alternative method for computing super-irreducible forms has been given in Levelt (1991).

**Proposition A2.** Let the system $x^{q+1}Y' = AY$ be super-irreducible and $\alpha$ defined as above. Then the change of variable $Y = x^{-\alpha}Z$ transforms the system into a system $x^{q+1}Z' = BZ$ which can be rewritten as a simple system.

**Proof.** The matrix $B$ is given by

$$B = x^\alpha (Ax^{-\alpha} + \alpha x^{q-\alpha}) = (x^\alpha N + \alpha)x^q.$$ Setting $D := x^{-\alpha}$ and multiplying by $Dx^{-q}$ on the left one obtains the matrix operator

$$\mathfrak{D} = D\theta - (N + \alpha D).$$

Its associated polynomial is (note that $\alpha D_0 = 0$)

$$P(\lambda) = \det(N_0 + \alpha D_0 - \lambda D_0) = \det(N_0 - \lambda D_0) \neq 0$$

since $\Theta_q(\lambda) = \det(N_0 + \lambda D_0) \neq 0$. Hence the operator $\mathfrak{D}$ is simple. □

Note that from an algorithmic viewpoint, this transformation is easy to compute.

**Example 5.** Consider again the system from Example 1. We showed that it can be immediately rewritten as a simple system. However, it is not super-irreducible. If one applies the super-reduction algorithm, one obtains the matrix

$$\begin{pmatrix}
-5 & 0 & 2x^{-1} \\
-19 + 30x^{-1} & 18 + x^{-1} & -102 \\
-3 + 5x^{-1} & 3 & -17
\end{pmatrix}$$

which is super-irreducible. It turns out that the corresponding system can be written as a simple system with corresponding matrices $\alpha = \text{diag}(-1, -1, -1)$, $D_0 = 0$ and

$$N_0 = \begin{pmatrix}
0 & 0 & 2 \\
30 & 1 & 0 \\
5 & 0 & 0
\end{pmatrix}.$$ Since $D_0 = 0$, the indicial polynomial is $P(\lambda) = \det(N_0) = -10$. Note that the resulting super-irreducible matrix is of lower pole order than the original matrix, and its coefficients are more complicated. In this case, it is more efficient to test first whether the system can be rewritten as a simple system before applying the super-reduction algorithm. The notion of simplicity is therefore weaker than the super-irreducibility.

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